Kepler-type dynamical symmetries of long-range monopole interactions

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A general framework for understanding Kepler-type dynamical symmetries is presented. The main concern is the geodesic motion in Euclidean Taub–NUT space, which approximates the scattering of self-dual monopoles for long distances. Other examples include a test particle moving in the asymptotic field of a self-dual monopole and two other related metrics.

I. INTRODUCTION

In this paper we present a general method for understanding Kepler-type dynamical symmetries. Our main interest lies in explaining those symmetries found recently in the long-distance limit of monopole–monopole scattering,[1–4] as well as for a test particle in the asymptotic field of a self-dual monopole.[5]

In the long-distance limit, the relative motion of two monopoles is approximately described in fact by the geodesics of the Euclidean Taub–NUT space of parameter $m = -\frac{1}{2}$, with the line element

$$ds^2 = (1 + 4m/r)(dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)) + [(4m)^2/(1 + 4m/r)](d\psi + \cos\theta\, d\theta)^2. \quad (1.1)$$

For $m > 0$, (1.1) is just (the space part of) the line element of the celebrated Kaluza–Klein monopole of Gross and Perry and Sorkin.[6] The problem of geodesic motion in this metric therefore has its own interest independent of monopole scattering. The relativistic aspects of such metrics have been studied recently by Gibbons and Ruback[3] in great detail. Here we explore instead the relation to *dynamical symmetries.*

In the Taub–NUT limit the electric charge q (the Nœther quantity conjugate to $\partial/\partial\psi$) is conserved and, for a fixed q, symplectic reduction[7] leads to the three-dimensional Hamiltonian

$$h = \tfrac{1}{2}(\mathbf{p}^2/(1 + 4m/r) + (1 + 4m/r)(q/4m)^2), \quad (1.2)$$

where $\mathbf{p} = (1 + 4m/r)\mathbf{v} = \partial\pounds/\partial\mathbf{v} - q\mathbf{A}$, where $\mathbf{A}$ is a Dirac monopole vector potential. The fundamental Poisson brackets $\{p_i,p_j\} = -q\epsilon_{ijk}(r^k/r^3)$, $\{r_i,p_j\} = \delta_{ij}$ correspond to the symplectic form

$$\Omega = d\mathbf{r} \wedge d\mathbf{p} + (q/2r^3)\epsilon_{ijk} r^i\, dr^j \wedge dr^k \quad (1.3)$$

on the phase space, according to $\{f,g\} = \Omega(X_f,X_g)$, $df = \Omega(X_f,\cdot)$.

The second system we study here is a *spinless test particle* moving *outside the core of a self-dual monopole.* The Higgs field Φ can be identified with the extra space component of a pure Yang–Mills field in four dimensions. The equations of motion can be obtained from the Kerner–Wong[8] equations in $1 + 4$ dimensions by dimensional reduction.[5] For large distances, the only surviving gauge field component is the one parallel to Φ. The particle's isospin projects into the conserved electric charge q.[9] This leaves us with an effective Dirac monopole and a long-range scalar field $\Phi \sim 1-1/r$. Our particle is described by the same symplectic structure as in (1.3) and the Hamiltonian

$$h = \tfrac{1}{2}(\mathbf{p}^2 + q^2[1 - 1/r]^2), \quad (1.4)$$

where $\mathbf{p}$ is the ordinary momentum. This system was studied previously by McIntosh and Cisneros and Zwanziger[10] because of its remarkable symmetries, but without its present physical interpretation. Such a role has been hinted at by Schönfeld.[11] See, also, Refs. 5 and 12.

For both systems, the clue of the solutions is provided by a conserved Runge–Lenz-type vector, which allows one[2,3] to prove that the trajectories are conic sections. We shall mainly consider the bound motions. We mainly concentrate on the more recent and less explored Taub–NUT problem.

Observing that the conserved angular momentum vector $\mathbf{j}$ and the rescaled Runge–Lenz vector $\mathbf{k}$ [(2.8)] form an o(4) algebra for the bound motions and an o(3,1) algebra for the scattered motions, the Pauli method[13] allows one to recover the bound-state spectrum and the Zwanziger method[10] allows one to derive the S matrix.[2]

The o(4)/o(3,1) symmetry can be extended into o(4,2). For example,[3] application of the so-called "Kustaanheimo–Stiefel"[14] transformation carries the Taub–NUT system into a *harmonic oscillator.* The latter admits an sp(8, $\mathbf{R}$) dynamical symmetry; those transformations that preserve the charge constraint form an su(2,2)$\approx$o(4,2).

In Barut's method[15] (for the Taub–NUT system, for example[4]), one starts instead with the *time-independent* Schrödinger equation $\hat{h}\Psi = e\Psi$. Assuming that $e < q^2/32m^2$, one multiplies the Schrödinger equation by $(r + 4m)$ and redefines position and momenta as

$$\mathbf{R} = \sqrt{(q/4m)^2 - 2e}\,\mathbf{r}, \quad \mathbf{P} = \mathbf{p}/\sqrt{(q/4m)^2 - 2e}. \quad (1.5)$$

After rearrangement, the Schrödinger equation takes the form

$$\{\tfrac{1}{2}R(\mathbf{P}^2 + 1) + q^2/2R\}\Psi = (4m[e - (q/4m)^2]/\sqrt{(q/4m)^2 - 2e})\Psi. \quad (1.6)$$

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On the lhs of (1.6) one recognizes Γ_0, the generator of an o(2,1) group, to which one can add[4,15] 14 more operators, cf. (3.11 a–g) which generate an o(4,2) operator algebra *independent* of the energy constraint. Therefore, the solution of the eigenvalue equation (1.6) can be deduced from the spectrum of Γ_0. The same procedure works in the other cases.[15]

The method we present here consists in completing $f:(\mathbf{r}, \mathbf{p}) \to (\mathbf{R},\mathbf{P})$ into a *canonical transformation*. We do this by unfolding the system into Souriau's[16] "espace d'évolution" (evolution space) $\mathscr{E} = M \times \mathbf{R}$, which is endowed with the presymplectic structure $\sigma = \Omega + dh \wedge dt$. [$\mathscr{E}$ can also be viewed as the seven-dimensional "energy surface" lying in the eight-dimensional extended phase space $T^*(\mathbf{R}^3 \times \mathbf{R})$[17].] The classical motions are the characteristic curves of σ. This is basically a generalized variational formalism[16]: If θ is a potential for σ, $d\theta = -\sigma$, then the classical action is $\int \pounds = \int \theta$.

The quotient $(\mathscr{N},\omega)$ of $(\mathscr{E},\sigma)$ by the characteristic foliation of σ is Souriau's "espace des mouvements" (space of motions).[16] In this framework, a symmetry is a transformation of $\mathscr{E}$ which preserves σ and thus permutes the classical motions: it projects into a symplectomorphism of $(\mathscr{N}, \omega)$.

The information on the global structure is encoded into the topology of $\mathscr{N}$. A fixed $t = t_0$ section N_0 of the evolution space is the "phase space at t_0"[16]; the restriction of σ to N_0 is symplectic. The mapping $N_0 \to \mathscr{N}$ (obtained by composing with the projection $\mathscr{E} \to \mathscr{N}$) is injective and symplectic, but may *not* be *onto*. In the Kepler problem, for example, the phase space N_0 does not intersect those motions that hit the center at $t = t_0$.[18] Therefore, N_0 may not reflect the global structure of the space of motions.

The situation is similar for the Taub–NUT system. The metric (1.1) is singular for $r = 4|m|$, which should be excluded. The energy is positive for $r > 4|m|$ and negative for $r < 4|m|$ and, by energy conservation, no motion can cross the singular sphere $S = \{r = 4|m|\}$. Hence the space of bound motions has two connected components. The negative-energy part $\mathscr{E}_-$ of the Taub–NUT evolution space $\mathscr{E}$ contains the *tightly-bound motions* ($\mathscr{N}_-$) and the positive-energy part $\mathscr{E}_+$ contains the *lightly-bound motions* ($\mathscr{N}_+$). For us, $\mathscr{E}_+$ is more interesting since the Taub–NUT approximation is justified only for large r.

In both components, the radial motions leave their regions and hit the singularity. In other words, for $m < 0$ the Taub–NUT space is not a complete Riemann manifold. Consequently, the spaces of motions $\mathscr{N}_\pm = \mathscr{E}_\pm/\mathrm{Ker}\,\sigma$ are not Hausdorff.

A regular system is one whose presymplectic form defines a foliation with one-dimensional, infinite curves: Its space of motions *is* a Hausdorff manifold. Regularizing the Taub–NUT problem requires imbedding it into a regular "unphysical" one by an injective, symplectic mapping f whose image is a dense, open subset. Those "unphysical" motions, that correspond to the Taub–NUT motions that leave the evolution space can be made infinite by restoring their points not in $\mathrm{Im}\, f$. Identifying the preimages in the Taub–NUT space of motions, we obtain a smooth Hausdorff manifold, namely the "unphysical" motion space.

We choose the following regular "unphysical" system $(\mathscr{M}_s, \Sigma_s)$: We consider in fact those zero-mass, helicity-s, coadjoint orbits (M_s,ϖ_s) associated with the action of SU(2,2) on twistor space.[19–21] The su(2,2) $\simeq$ o(4,2) generators are the classical counterparts of those operators in Refs. 4, 15, 21, and 22. Choosing the generator Γ_0 [(3.12a)] as Hamiltonian and adding a "fake time" T, we obtain an "unphysical" evolution space $\mathscr{M}_s = M_s \times \mathbf{R}$ endowed with $\Sigma_s = \varpi_s + dH \wedge dT$. The o(4,2) generators are extended to $\mathscr{M}_s$ so as to remain constant along the trajectories. The space of "unphysical" motions, $\mathscr{M}_s/\mathrm{Ker}\,\Sigma_s$, is *globally symplectomorphic* to the $T = 0$ phase space which is (M_s,ϖ_s). This system has a *manifest* $C^{\uparrow}_{+}(3,1) \simeq \mathrm{SU}(2,2)/(\mathrm{center})$ *symmetry*.

To summarize, our canonical transformation f allows us to *regularize* the "physical" problem as well as exhibit its "hidden" conformal symmetry. Our method is particularly useful in discussing global problems.

This transformation is found by completing (1.5) with the rule of transforming the time,

$$T = \left[\sqrt{(q/4m)^2 - 2h}/4mh\right] \times \{-\mathbf{p}\cdot\mathbf{r} - ((q/4m)^2 - 2h)t\}. \tag{1.7}$$

Equation (1.7) is chosen to compensate for the noninvariance of $d\mathbf{r} \wedge d\mathbf{p}$ under (1.5), due to the energy being a function rather than a constant. The pullback of the "unphysical" presymplectic form is the Taub–NUT presymplectic form. (The Lagrangians differ by a total derivative.)

The regularized lightly-bound Taub–NUT motion spaces $\mathscr{N}_+$ are thus shown to be symplectomorphic to (M_s,ϖ_s). The same is true for the tightly-bound motions $\mathscr{N}_-$. Therefore, both carry a $C^{\uparrow}_{+}(3,1)$ conformal symmetry.

In the McIntosh–Cisneros (MIC)–Zwanziger case no regularization is necessary and M_s is thus also the space of bound test-particle motions in the asymptotic monopole field. (M_0 is the space of regularized motions of the Kepler problem.[21] Our method also yields the $C^{\uparrow}_{+}(3,1)$ symmetry.[12]

The space of twistors can also be viewed as the phase space of a four-dimensional harmonic oscillator from which the "unphysical" system is obtained by reduction.[3,15,22]

In Sec. V we study the scattered motions. We show that the space of regularized hyperbolic motions is symplectomorphic to the orbit (M_s, ϖ_s) and hence carries (unlike in the Kepler case[18]) an action of the conformal *group*.

We end this paper by a short discussion of two other (closely related) metrics whose geodesics are also o(4,2) symmetric. The first metric (which is new) can be viewed as a curved-space model for a particle in a self-dual monopole field and the other has been found recently in describing some special motions of a closed string in the Taub–NUT background.

Applied to the Kepler problem, our method would yield an imbedding into M_0, which is the standard regularization,[14,18,23] since $M_0 \simeq T^+(\mathrm{S}^3) = T^*(\mathrm{S}^3)\setminus(\text{zero section})$. The conformal symmetry is obtained for free.

II. CLASSICAL MOTIONS IN TAUB–NUT

Neglecting radiation, the motion of two self-dual monopoles is approximately described by the geodesics in the space of solutions of the Bogomolny equation, called the moduli space.[1] The moduli space is the product of $\mathbb{R}^3\times S^1$, the manifold of the center-of-mass motion, with a curved four-manifold whose metric was found by Atiyah and Hitchin.[1] The latter describes the relative motion of the monopoles. In the long-distance limit exponential terms can be neglected and we obtain a Euclidean Taub–NUT space of parameter $m = -\frac{1}{2}$, with the line element (1.1). The geodesic motion of a spinless particle of unit mass in (1.1) is described by the Lagrangian

$$\pounds = \tfrac{1}{2} g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = \tfrac{1}{2}\big((1+4m/r)\mathbf{v}^2 + [(4m)^2/(1+4m/r)](\dot\psi+\cos\theta\dot\phi)^2\big), \tag{2.1}$$

where $\mathbf{v}=\dot{\mathbf{r}}$. Here $r>0$ and the angles θ, ϕ, ψ ($0\leqslant\theta\leqslant\pi$, $0\leqslant\phi<2\pi$, $0\leqslant\psi<4\pi$) parametrize S^3. The points $r=4|m|$, where the metric (1.1) is singular, are excluded. The conserved Noether quantity

$$q=(4m)^2[(\dot\psi+\cos\theta\dot\phi)/(1+4m/r)] \tag{2.2}$$

associated to the cyclic variable ψ is the *relative electric charge*. From now on we choose and fix a nonzero value for q. It is convenient to introduce the "mechanical momentum" $\mathbf{p}=(1+4m/r)\mathbf{v}$. The equation of motion is then

$$\frac{d\mathbf{p}}{dt} = -2m\frac{\mathbf{v}^2}{r^3}\mathbf{r} + \frac{q^2}{8m}\frac{\mathbf{r}}{r^3} - q\frac{\mathbf{v}\times\mathbf{r}}{r^3}. \tag{2.3}$$

We have the following conserved quantities. First, the energy,

$$e=\tfrac{1}{2}(1+4m/r)[\mathbf{v}^2+(q/4m)^2], \tag{2.4}$$

and next the monopole angular momentum

$$\mathbf{j}=\mathbf{r}\times\mathbf{p}+q(\mathbf{r}/r). \tag{2.5}$$

Finally, we have the Runge–Lenz-type vector

$$\mathbf{a}=\mathbf{p}\times\mathbf{j}-\frac{\mathbf{r}}{r}(4m(e-(q/4m)^2)). \tag{2.6}$$

Hence the trajectories lie simultaneously on the cone $\mathbf{j}\cdot\mathbf{r}/r=q$ and also in the plane perpendicular to

$$\mathbf{n}=q\mathbf{a}+4m[e-(q/4m)^2]\mathbf{j}, \tag{2.7}$$

because of the relation $\mathbf{n}\cdot\mathbf{r}=q(j^2-q^2)$. They are thus conic sections.[1,2]

For energies smaller than $(q/4m)^2/2$ (which is only possible for $m<0$) the motions are bound. We assume henceforth that $m<0$.

Under the Poisson bracket and for $e<(q/4m)^2/2$, the angular momentum $\mathbf{j}$ closes, with the rescaled Runge–Lenz vector

$$\mathbf{k}=\frac{\mathbf{a}}{\sqrt{|(q/4m)^2-2e|}} = \frac{\mathbf{p}\times\mathbf{j}-(\mathbf{r}/r)(4m(e-(q/4m)^2))}{\sqrt{|(q/4m)^2-2e|}}, \tag{2.8}$$

into an o(4) dynamical symmetry algebra. For $e>(q/4m)^2/2$ we instead obtain an o(3,1) algebra.[2]

Via (2.4), the sign of the energy depends on r being smaller or larger than $4|m|$. The excluded points form a singular sphere

$$S=\{\mathbf{r}\in\mathbb{R}^3 | r=4|m|\}, \tag{2.9}$$

which divides the space into two regions. Energy conservation implies that a particle cannot go from one region into the other (although it can hit the boundary S, see below). If a finite-energy motion approaches S, its velocity $|v|$ goes, via (2.4), to infinity as $(1-4|m|/r)^{-1/2}$; its momentum $\mathbf{p}$ hence goes to zero as $(1-4|m|/r)^{1/2}$. Those motions in the interior of S have negative energy; they are the *tightly-bound motions* $\mathcal{N}_-$. Those motions in the exterior and having energy $0<e<q^2/32m^2$ are the *lightly bound motions* $\mathcal{N}_+$. We shall focus our attention on $\mathcal{N}_+$.

In the generic case the orbital angular momentum $\mathbf{r}\times\mathbf{p}$ is nonzero, and the cone has opening angle $\alpha(\cos\alpha=|q|/|\mathbf{j}|)$. Such motions avoid S. Indeed, we see from (2.5) that for nonvanishing orbital angular momentum, $\mathbf{j}$ cannot be radial. However, when hitting the singular sphere S, the orbital part necessarily vanishes requiring $\mathbf{j}$ to be radial.

Consider now the radial motions. Fixing a direction, we work with r, p. If the initial velocity is inward, the particle reaches the singularity in finite time, and leaves the "physical" space. If the velocity is outward (but sufficiently low as to remain bound),

$$v=\sqrt{((2e-(q/4m)^2)r+q^2/4|m|)/(r+4m)} \tag{2.10}$$

shows that there will be a unique turning point $r_1>0$ where v vanishes, namely at

$$r_1=4|m|\{(q/4m)^2/[(q/4m)^2-2e]\}>4|m|. \tag{2.11}$$

After reaching r_1, the particle returns and falls inward until it disappears in S. At this very moment, another radial motion leaves the singularity and follows the same phase-space trajectory backward. When passing to the quotient, any two neighborhoods of these two motions intersect. In order to obtain a Hausdorff topology, such motions should—and will—be identified. All motions then become periodic.

The set of outer turning points of radial motions is

$$B^0\times\mathbb{R}=\{(\mathbf{r},\mathbf{p},t)|\ |\mathbf{r}|>4|m|,\mathbf{p}=0\} \tag{2.12}$$

and the set of inner turning points is

$$S\times\mathbb{R}=\{(\mathbf{r},\mathbf{p},t)|\ |\mathbf{r}|=4|m|,\mathbf{p}=0\}. \tag{2.13}$$

The situation is basically the same for the tightly-bound motions. The nonradial motions are ellipses which avoid the origin as well as S. A radial motion has an internal turning point at $0<r_1<4|m|$, according to (2.11). All radial motions fall into S in finite time from the inside, with infinite velocity and zero momentum: Such a motion should be identified with the motion that leaves S in the opposite direction along the same trajectory.

III. SOME MANIFESTLY o(4,2)-SYMMETRIC SYSTEMS

A twistor[19,20] can be represented by a pair of spinors $Z^\alpha=(\omega^A,\pi_{A'})$ in $\mathbb{T}=(\mathbb{C}^2\times\mathbb{C}^2)\setminus\{0\}$. (Here $\pi_{A'}$ plays the role of a generalized coordinate and ω^A plays that of a generalized momentum.) The conjugate of $Z^\alpha(\alpha=0,1,2,3)$ is $Z^*_\alpha=(\pi^*_A,(\omega^*)^{A'})=((\pi_{A'})^*,(\omega^A)^*)$ (the asterisk means complex conjugate). The space of twistors is endowed with a Hermitian quadratic form of signature (2,2) given by

$$Z^\alpha Z^*_\alpha = \omega^A \pi^*_A + \pi_{A'}(\omega^*)^{A'}, \quad A=0,1; A'=0',1'. \quad (3.1)$$

To each real number s we associate a (real) seven-dimensional manifold $\mathbf{T}_s$, namely the level surface

$$Q(Z^\alpha, Z^*_\alpha) = \tfrac{1}{2} Z^\alpha Z^*_\alpha = s. \quad (3.2)$$

Here $\mathbf{T}$ carries the (by construction) U(2,2) invariant one-form

$$\theta = -(i/2)(Z^\alpha dZ^*_\alpha - Z^*_\alpha dZ^\alpha) \quad (3.3)$$

whose exterior derivative

$$-d\theta = i\, dZ^\alpha \wedge dZ^*_\alpha \quad (3.4)$$

is a symplectic form on $\mathbb{T}$. The Poisson brackets are thus $\{Z^\alpha, Z^*_\beta\} = -i\delta^\alpha_\beta$. The restriction ϖ_s of $(-d\theta)$ to the level surface $\mathbf{T}_S$ defines a one-dimensional integrable foliation and ϖ_s descends to M_s, the quotient of $\mathbf{T}_s$ by the characteristic foliation of ϖ_s. In this way M_s becomes a six-dimensional symplectic manifold. Explicitly, the characteristic curves of ϖ_s (the Hamiltonian flow of Q) are circles,

$$Z^\alpha \to e^{-i\rho/2} Z^\alpha, \quad Z^*_\alpha \to e^{i\rho/2} Z^*_\alpha, \quad 0 \leqslant \rho \leqslant 4\pi, \quad (3.5)$$

which identifies M_s as $\mathbf{T}_s/\mathrm{U}(1)$.

The unitary group U(2,2) leaves invariant the quadratic form (3.1) and thus, also, the level surfaces $\mathbf{T}_s$. The action of U(2,2) on $\mathbf{T}_s$ is clearly transitive. The action of the diagonal U(1) subgroup of U(2,2) on $\mathbf{T}_s$ coincides with the flow (3.5). Therefore, it is only SU(2,2) that acts on the quotient. In this way we obtain a transitive, symplectic action of SU(2,2) on (M_s, ϖ_s). Souriau's moment map[16] therefore identifies (M_s, ϖ_s) with a coadjoint orbit of SU(2,2), endowed with its canonical symplectic structure. M_s can also be viewed as a U(2,2) coadjoint orbit, where s is an element in the center of the Lie algebra.

For $s \neq 0$ the Poincaré subgroup of SU(2,2) already acts transitively, so that M_s is actually sumplectomorphic to the Poincaré orbit ($\mathscr{Q}_{0,s,+}$), the space of motions of a relativistic, zero-mass, helicity-s, elementary particle.

For $s=0$ the action of the Poincaré subgroup on M_0 is no longer transitive and M_0 is rather the space of motions of a helicity-zero, mass-zero particle in *compactified* Minkowski space $\mathbf{S}^1 \times \mathbf{S}^3$. M_0 is obtained from the zero-mass Poincaré orbit ($\mathscr{Q}_{0,0,+}$) by adding those motions that lie along the generators of the light cone at infinity.

As will be clear from the parametrization below, all zero-mass Poincaré orbits are diffeomorphic to $\mathbb{R}^3 \times (\mathbb{R}^3 \setminus \{0\})$. This is thus the topology of M_s for $s \neq 0$. The topology of M_0 is, in turn, $\mathbf{S}^3 \times (\mathbb{R}^3 \setminus \{0\})$. Indeed,

$$R^\mu = \sigma^\mu_{AA'} \pi^*_A \pi_{A'}, \quad (3.6)$$

where the σ_μ are the Pauli matrices, determines, for any $\pi_{A'} \in \mathbb{C}^2 \setminus \{0\}$, a unique future-pointing light-like vector $(R^\mu) = (R, \mathbf{R})$ ($R = |\mathbf{R}|$) in Minkowski space. Conversely, those π's that solve Eq. (3.6) for a given $\mathbf{R}$ belong to a circle. This is clear from the following:

$$\pi_{A'} = \sqrt{R} \begin{pmatrix} \cos(\theta/2) e^{-i(\psi+\phi)/2} \\ \sin(\theta/2) e^{i(-\psi+\phi)/2} \end{pmatrix}. \quad (3.7)$$

The vector $\mathbf{R}$ has the polar coordinates R, θ, ϕ. The map $\pi_{A'} \to \mathbf{R}$ is thus essentially the projection of the Hopf fibering $\mathbf{S}^3 \to \mathbf{S}^2$; its (multivalued) inverse is the Kustaanheimo–Stiefel[14] transformation.

Choosing $\pi_{A'}$ to a pair $(\mathbf{P}, \mathbf{R})$ in $\mathbb{R}^3 \times (\mathbb{R}^3 \setminus \{0\})$ we can associate a twistor $Z^\alpha = (\omega^A, \pi_{A'})$ by setting

$$\omega^A = i(P^k \sigma_k{}^{AA'} - i(s/R)\sigma_0{}^{AA'})\pi_{A'}. \quad (3.8)$$

For any choice of $\pi_{A'}$ (i.e., of the phase ψ) Z^α belongs to $\mathbf{T}_s$ and the whole of $\mathbf{T}_s$ is obtained. Thus the pairs $(\mathbf{P}, \mathbf{R})$ parametrize those circles in Eq. (3.5) and thus the quotient manifold M_s.

For $s=0$, $\mathbf{T}_0 = \mathbf{T}^0_0 \cup \mathbf{T}^0_\infty$, $(\mathbf{T}^0_0 \cap \mathbf{T}^0_\infty = 0)$, where $\mathbf{T}^0_0 = \{(\omega^A, \pi_{A'}) \in \mathbf{T}_0 | \pi_{A'} \neq 0\}$ and $\mathbf{T}^0_\infty = \{(\omega^A, \pi_{A'}) | \omega^A \neq 0, \pi_{A'} = 0\}$. The complex projective lines in $\mathbf{PT}$ meeting $\mathbf{PT}^0_\infty$ corresponds to points at infinity in (compactified and complexified) Minkowski space. Therefore, the orbit M_0 is decomposed as

$$M_0 = \mathscr{O}_{0,0,+} \cup M^0_\infty, \quad \mathscr{O}_{0,0,+} = \mathbf{T}^0_0/\mathrm{U}(1),$$
$$M^0_\infty = \mathbf{T}^0_\infty/\mathrm{U}(1). \quad (3.9)$$

As anticipated by the notation, $\mathscr{O}_{0,0,+}$ is a zero-mass, zero-helicity Poincaré orbit because the Poincaré subgroup of SU(2,2), leaves the constraint $\pi_{A'} \neq 0$ invariant. Here M^0_∞ describes those motions that lie along the generators of the light cone at infinity. The decomposition (3.9) also shows that M_0 is symplectomorphic to $T^+\mathbf{S}^3$, the cotangent bundle of the three-sphere with its zero section deleted. Indeed, $\mathbb{R}^3$ is (by stereographic projection) $\mathbf{S}^3$ without its north pole; the Poincaré orbit $\mathscr{O}_{0,0,+}$ is $\mathbb{R}^3 \times (\mathbb{R}^3 \setminus \{0\}) \simeq T^+(\mathbf{S}^3 \setminus \{N\})$ and $M^0_\infty \simeq T^+_N(\mathbf{S}^3)$.

The action of $\mathrm{su}(2,2) \approx \mathrm{o}(2,4)$ on $\mathbb{T}$ is generated by the matrices

$$\gamma_{0k} = -\frac{i}{2}\begin{pmatrix} \sigma_k & 0 \\ 0 & -\sigma_k \end{pmatrix}, \quad \gamma_{jk} = \frac{1}{2}\epsilon_{jkn}\begin{pmatrix} \sigma_n & 0 \\ 0 & \sigma_n \end{pmatrix},$$
$$\gamma_{06} = \frac{1}{2}\begin{pmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{pmatrix}, \quad \gamma_{k6} = \frac{1}{2}\begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix},$$
$$\gamma_{05} = \frac{1}{2}\begin{pmatrix} 0 & \sigma_0 \\ -\sigma_0 & 0 \end{pmatrix}, \quad \gamma_{k5} = \frac{1}{2}\begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix},$$
$$\gamma_{56} = \frac{i}{2}\begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix},$$
$$(\gamma_{LK} = -\gamma_{KL}, \quad K,L = 0,\ldots,3; 5, 6). \quad (3.10)$$

[Our convention for the metric on $\mathbb{R}^{2,4}$ is $g_{KL} = \mathrm{diag}(g_{\mu\mu}, g_{55}, g_{66}) = (+1,-1,-1,-1;-1,+1)$.] The matrices in (3.10) leave invariant the quadratic form (3.1) and the symplectic form (3.4). The components of the moment map are $J_{KL} = Z^*_\alpha (\gamma_{KL})^\alpha_\beta Z^\beta$. In dynamical group notations,[15] on each orbit we have the 15 generators

$$J_{06} \to \Gamma_0 = R[(\mathbf{P}^2+1)/2] + s^2/2R, \quad (3.11a)$$
$$\tfrac{1}{2}\epsilon_{ijk} J_{jk} \to \mathbf{J} = \mathbf{R} \times \mathbf{P} + s(\mathbf{R}/R), \quad (3.11b)$$
$$J_{5i} \to \mathbf{K} = \mathbf{R}\frac{\mathbf{P}^2-1}{2} - \mathbf{P}(\mathbf{R}\cdot\mathbf{P}) - \frac{s}{R}\mathbf{J} + s^2 \frac{\mathbf{R}}{2R^2}$$
$$= \mathbf{P} \times \mathbf{J} - (\mathbf{R}/R)\Gamma_0, \quad (3.11c)$$

$$J_{i6}\to \mathbf{U} = \mathbf{R}\frac{\mathbf{P}^2+1}{2} - \mathbf{P}(\mathbf{R}\cdot\mathbf{P}) - \frac{s}{R}\mathbf{J} + s^2\frac{\mathbf{R}}{2R^2}$$

$$= \mathbf{P}\times\mathbf{J} - (\mathbf{R}/R)\Gamma_4, \quad (3.11d)$$

$$J_{56}\to D = -\mathbf{R}\cdot\mathbf{P}, \quad (3.11e)$$

$$J_{i0}\to \mathbf{V} = -R\cdot\mathbf{P}, \quad (3.11f)$$

$$J_{50}\to \Gamma_4 = R\frac{\mathbf{P}^2-1}{2} + \frac{s^2}{2R}. \quad (3.11g)$$

In particular, Γ_0, Γ_4, and D generate an o(2,1) subalgebra; those generators that commute with Γ_0 are $\mathbf{J}$ and $\mathbf{K}$, which form an o(4) subalgebra—the "invariance algebra" of the Hamiltonian Γ_0. The remaining o(4,2) generators are sometimes called "noninvariance" generators.

From (3.11) we see that

$$\mathbf{R} = \mathbf{U} - \mathbf{K}, \quad \mathbf{P} = -\mathbf{V}/R; \quad (3.12)$$

thus from the o(2,4) relations $\{J_{KL},J_{MN}\} = g_{KN}J_{LM} + g_{LM}J_{KN} - g_{KM}J_{KL} - g_{LN}J_{KM}$ we derive the symplectic form ϖ_s of $\mathscr{O}_{0,s,+}$:

$$\varpi_s = dR_i\wedge dP_i + (s/2R^3)\epsilon_{ijk}R^i dR^j\wedge dR^k. \quad (3.13)$$

Now we construct a classical dynamical system which has a *manifest* SU(2,2) symmetry. Consider, in fact, the evolution space

$$\mathscr{M}_s = M_s\times\mathbb{R} = \{\mathbf{R},\mathbf{P},T\}, \quad \Sigma_s = \varpi_s + dH\wedge dT, \quad (3.14)$$

where ϖ_s is the symplectic form of the orbit M_s and the Hamiltonian is

$$H(\mathbf{R},\mathbf{P}) = \Gamma_0(\mathbf{R},\mathbf{P}). \quad (3.15)$$

Let us extend the 15 generators of o(4,2) in (3.11) to $\mathscr{M}_s$ such that they remain conserved along the trajectories:

$$H^{\sim} = H = \Gamma_0, \quad (3.16a)$$

$$\mathbf{J}^{\sim} = \mathbf{J}, \quad (3.16b)$$

$$\mathbf{K}^{\sim} = \mathbf{K}, \quad (3.16c)$$

$$U_\alpha^{\sim} = U_\alpha\cos T + V_\alpha\sin T, \quad \alpha = 1,2,3,5, \quad (3.16d)$$

$$V_\alpha^{\sim} = -U_\alpha\sin T + V_\alpha\cos T, \quad \alpha = 1,2,3,5, \quad (3.16e)$$

where we have introduced the "Bacry–Györgyi"[24] four-vectors $(U_\alpha) = (\mathbf{U},D)$ and $(V_\alpha) = (\mathbf{V},\Gamma_4)$.

Combining (3.12) with (3.16) yields an explicit integration of the equations of motion:

$$\mathbf{R}(T) = \mathbf{U}^{\sim}\cos T - \mathbf{V}^{\sim}\sin T - \mathbf{K},$$

$$\mathbf{P}(T) = -(\mathbf{U}^{\sim}\sin T + \mathbf{V}^{\sim}\cos T)/R(T). \quad (3.17)$$

Equations (3.17) show that the orbits are ellipses, with period $\Delta T = 2\pi$. The Runge–Lenz vector $\mathbf{K}$ points from the origin into the center of the ellipse. The orbit is the intersection of the cone $\mathbf{R}\cdot\mathbf{J} = s$, with the plane normal to the vector

$$\mathbf{N} = -\mathbf{U}^{\sim}\times\mathbf{V}^{\sim} = s\mathbf{K} + \Gamma_0\mathbf{J}. \quad (3.18)$$

The quotient of $(\mathscr{M}_s, \Sigma_s)$ by the characteristic foliation of Σ_s is the space of "unphysical" motions. Since every motion is infinite and depends regularly on the initial conditions, this quotient is *globally symplectomorphic* to the $T=0$ phase space, which is the SU(2,2) orbit (M_s,ϖ_s). The projection $\pi:\mathscr{M}_s\to M_s$ maps a point $(\mathbf{R}_0, \mathbf{P}_0, T_0)$ into the point at $T=0$ on the unique classical motion through $(\mathbf{R}_0, \mathbf{P}_0, T_0)$.

The group SU(2,2) acts on $\mathscr{M}_s = M_s\times\mathbf{R}$ by acting on the first factor alone, without changing T. The center of SU(2,2) acts on the coadjoint orbit M_s trivially. Therefore, the Lie algebra action (3.16) integrates into a global symplectic action of the adjoint group of SU(2,2), which is the *conformal group* $C^{\uparrow}_{+}(3,1)$.

Notice that the "unphysical" energy function $H = \Gamma_0$ *satisfies* $H \geqslant |s|$ and equality is only achieved for $R = |\mathbf{R}| = |s|$ and $\mathbf{P} = 0$. A particle with the initial conditions $|\mathbf{R}(0)| = |s|$, $\mathbf{P}(0) = 0$ is in equilibrium.

Let us consider a motion with the initial conditions $\mathbf{R}(0) = \mathbf{R}_0$, $\mathbf{P}(0) = 0$ at $T=0$. Since now $\mathbf{U}^{\sim} = (\mathbf{R}_0/R_0)\Gamma 4^{\sim}$, $\mathbf{V}^{\sim} = 0$, $-\mathbf{K}^{\sim} = (\mathbf{R}_0/R_0)\Gamma_0$ the motion oscillates on a line segment between the turning points $\Gamma_0 \pm \Gamma_4^{\sim}$ according to (3.17):

$$\mathbf{R}(T) = (\mathbf{R}_0/R_0)(\Gamma_0 - \Gamma_4^{\sim}\cos T),$$

$$\mathbf{P}(T) = -[\tilde{\mathbf{U}}/|\mathbf{R}(T)|]\sin T. \quad (3.19)$$

Notice that

$$\Sigma\times\mathbb{R} = \{0 < R \leqslant |s|, P = 0, T\}, \quad \text{resp.},$$

$$\Xi\times\mathbb{R} = \{R > |s|, P = 0, T\} \quad (3.20)$$

are the sets of the inner, resp., outer turning points.

IV. REGULARIZATION

The classical flow of the Taub–NUT evolution space is not complete: The radial motions leave it. Simply adding $S\times\mathbb{R}$ would not solve the problem since from the points of $S\times\mathbb{R}$ infinitely many motions start, all with zero momentum. Therefore, we regularize by relating the Taub–NUT problem to the regular "unphysical" dynamical system of Sec. III. Let us first study the lightly bound case.

Our guiding principle is that the "hidden" o(4) symmetry generators $\mathbf{j}$ and $\mathbf{k}$ of Taub–NUT should go into the manifest o(4) symmetry generators $\mathbf{J}$ and $\mathbf{K}$ of the "unphysical" problem. This is achieved by setting $s = q$ ($\neq 0$) and defining $f(\mathbf{r},\mathbf{p},t) = (\mathbf{R},\mathbf{P},T)$, where

$$\mathbf{R} = \sqrt{(q/4m)^2 - 2h}\,\mathbf{r}, \quad \mathbf{P} = \mathbf{p}/\sqrt{(q/4m)^2 - 2h},$$

$$T = [\sqrt{(q/4m)^2 - 2h}/4mh]$$

$$\times(-\mathbf{p}\cdot\mathbf{r} - ((q/4m)^2 - 2h)t), \quad (4.1)$$

and h is the Taub–NUT Hamiltonian (1.2). The first two of Eqs. (4.1) ensure that f intertwines the o(4) generators and the last makes f canonical:

$$f^*\Sigma_s = f^*(\varpi_s + dH\wedge dT) = \Omega + dh\wedge dt = \sigma, \quad (4.2)$$

where

$$H = 4m\{[h - (q/4m)^2]/\sqrt{(q/4m)^2 - 2h}\}. \quad (4.3)$$

Expressing through the new variables $\mathbf{R}$ and $\mathbf{P}$ shows that H is the generator Γ_0 in (3.11a), which we have chosen for the "unphysical" Hamiltonian.

Now $f:(\mathbf{r},\mathbf{p},t)\to(\mathbf{R},\mathbf{P},T)$ maps the positive-energy Taub–NUT evolution space $\mathscr{E}_+$ into the "unphysical" evolution space $\mathscr{M}_s = M_s\times\mathbb{R}$. The formal inverses are

$(\mathbf{r},\mathbf{p},t) = f^{-1}(\mathbf{R},\mathbf{P},T)$, with

$$\mathbf{r} = 4|m|\,\frac{\mathbf{R}}{H \mp \sqrt{H^2 - s^2}}, \quad \mathbf{p} = \frac{H \mp \sqrt{H^2 - s^2}}{4|m|}\,\mathbf{P},$$

$$t = \left(\frac{4m}{H \mp \sqrt{H^2 - s^2}}\right)^2 \left\{\left(H + \frac{s^2}{H \mp \sqrt{H^2 - s^2}}\right)T - \mathbf{P}\cdot\mathbf{R}\right\}, \tag{4.4}$$

where $q = s$ and the energy transforms according to

$$h = [\sqrt{H^2 - s^2}/(4m)^2](\pm H - \sqrt{H^2 - s^2}). \tag{4.5}$$

In order to obtain a positive sign for h, we have to choose the upper signs.

Clearly, f cannot be a symplectomorphism because $(\mathscr{E}_+,\sigma)$ is not complete, while $(\mathscr{M}_s, \Sigma_s)$ is complete. In fact, $f = f|\mathscr{E}_+$ is not onto—but this is what we need. Denote $M_s \setminus \Sigma$ by M_s^0, where Σ is given in (3.20).

Proposition: Consider the dense, open subset $\mathscr{M}_s^0 = (M_s \setminus \Sigma) \times \mathbb{R}$ of the unphysical evolution space. Then (i) $f:(\mathscr{E}_+,\sigma) \to (\mathscr{M}_s^0, \Sigma_s)$ is a (pre)symplectic bijection and (ii) the inverse (4.4) extends naturally into a continuous mapping $\mathscr{M}_s \to \mathscr{E}_+ \cup (S \times \mathbb{R})$. Here f^{-1} carries the whole $\Sigma \times \mathbb{R}$ into the singularity $S \times \mathbb{R}$.

Proof: Since f preserves directions, it is sufficient to work with the absolute values $r = |\mathbf{r}|$, $p = |\mathbf{p}|$, $R = |\mathbf{R}|$, and $P = |\mathbf{P}|$. Also, since T depended on t linearly, we can—and will—drop the time variables when studying the global properties of f.

We first show that $\operatorname{Im} f$ does not contain those points $\{0 < R \leqslant |q|, P = 0\}$, i.e., the subset $\Sigma \subset M_s$. Indeed, $P = 0$ requires $p = 0$. Then, by (4.1), $R = |q|(r/4|m|)^{1/2} > |q|$ since $r > 4|m|$. Thus $\operatorname{Im} f \subset \mathscr{M}^0$. In order to prove that $\operatorname{Im} f$ fills $\mathscr{M}^0$, it is convenient to introduce some more points (cf. Fig. 1):

$$A' = (r = \infty, p = 0), \quad A = (R = \infty, P = 0),$$
$$C = (R = 0, P = \infty),$$

and we set $B^0 = \{(r > 4|m|, p = 0\}$ [cf. (2.11)] and $B^1 = \{(r, p(r))\}$, where $p(r) = (|q|/\sqrt{4|m|})\sqrt{1/r + 4m/r^2}$. Here B^0 belongs to $\mathscr{E}_+$, but B^1 does not. As we have seen, the interior points of the region whose boundary is $\{S\} \cup \{A'\} \cup B^1$ are carried by f into $\mathscr{M}^0$ and $f(A') = A$ $f(B^0) = \Xi$, $f(B^1) = C$.

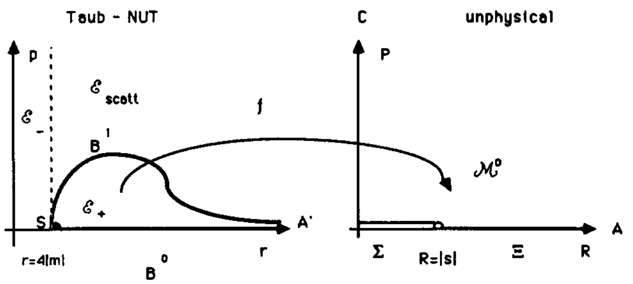

FIG. 1. The canonical transformation f in (4.1) takes the lightly-bound Taub–NUT evolution space $\mathscr{E}_+$ symplectically onto $\mathscr{M}^0 = (M_s \setminus \Sigma) \times \mathbb{R}$. The image of the entire $\Sigma \times \mathbb{R}$ by the inverse f^{-1} is the singularity $S \times \mathbb{R}$. Similarly, the tightly-bound evolution space $\mathscr{E}_-$ is carried into $(M_s \setminus \Xi) \times \mathbb{R}$ and now $f^{-1}(\Xi \times \mathbb{R}) = S \times \mathbb{R}$. The unbound part $\mathscr{E}_{scatt}$ is symplectomorphic to the full $\mathscr{M}$. Only the absolute values are shown and the time variables are dropped.

Let a be an arbitrary non-negative number and let us consider the hyperbolas $\mathscr{H}'_a = \{r \cdot p = a\}$ and $\mathscr{H}_a = \{R \cdot P = a\}$. The Taub–NUT evolution space is clearly the boundary B^0 plus the union of its intersections with the hyperbolas $\mathscr{H}'_a$. In turn, $\mathscr{M}^0$ is Ξ plus the union of the hyperbolas $\mathscr{H}^a$. Each hyperbola intersects the "upper boundary" B^1 at exactly one point, which is sent into C. Furthermore, the hyperbola $\mathscr{H}'_a$ is carried into $\mathscr{H}_a$ since $R \cdot P = r \cdot p$. It follows that $f(\mathscr{H}'_a \cap \mathscr{E}_+) = \mathscr{H}_a$. Adding the bottom line B^0 whose image is Ξ, we conclude that the image of f is the *entire* $\mathscr{M}^0$.

Finally, f is injective: A point (R,P) in Ξ is the image of $(r = 4|m|(R/s)^2, p = 0)$ from B^0; otherwise it lies on a unique hyperbola $\mathscr{H}_a$ and thus has a unique preimage in $\mathscr{H}'_a \cap \mathscr{E}_+$. This proves (i) of the proposition.

To prove (ii) of the proposition, observe that (4.4) is naturally defined for any point of $\mathscr{M}$. However, for a point in $\Sigma \times \mathbb{R}$, $H = R/2 + s^2/2R$, so that $H - (H^2 - s^2)^{1/2} = R$ since $R/2 - s^2/2R$ is negative for $R \leqslant |s|$. From (4.4) we infer that

$$f^{-1}(\Sigma \times \mathbb{R}) = (4|m|, 0, \mathbb{R}),$$

which is in the boundary $S \times \mathbb{R} = \{r = 4|m|, p = 0\} \times \mathbb{R}$ and does not belong to $\mathscr{E}_+$. The extension f^{-1} is clearly many-to-one. Q.E.D.

Regularizing $\operatorname{Im} f \subset \mathscr{M}_s$ is trivial: It is sufficient to add those turning points that we have excluded, i.e., $\Sigma \times \mathbf{R}$ [cf. (3.20)]. For Taub–NUT this amounts to gluing together the branches of the radial motions. When passing to the space of motions, this means identifying those points that thus far represent different (not infinite) motions and whose neighborhoods are not separated. This procedure yields a smooth, Hausdorff topology, namely that of M_s. To summarize, we present the following theorem.

Theorem: The map f: $(\mathscr{E}_+,\sigma) \to (\mathscr{M}_s, \Sigma_s)$ regularizes the Taub–NUT problem: It intertwines the time-independent o(4) symmetries. Here $(\mathscr{N}_+, \omega)$, the space of regularized lightly-bound Taub–NUT motions, is symplectomorphic to the SU(2,2) orbit (M_s, ϖ_s) and hence carries a symplectic action of the conformal group $C^{\uparrow}_+(3,1)$.

The results in Sec. II are consequences of what we have found in Sec. III and the properties of the canonical transformation f. For example, it follows from (4.1) and (3.17) that the trajectories are ellipses in the plane perpendicular to the vector n in (2.7),

$$\mathbf{n} = q\mathbf{k} + 4m\{[e - (q/4m)^2]/\sqrt{(/4m)^2 - 2e}\}\mathbf{j}, \tag{4.6}$$

which is (up to normalization) the image of $\mathbf{N}$ in (3.18).

The pullbacks of the 15 generators in (3.16) by f yield o(4,) symmetry generators of the Taub–NUT system: They coincide with the classical counterparts of the quantum operators written in Ref. 4. *Without* regularization, this would only yield an o(4,2) *algebra*.

The period of a Taub–NUT motion could be obtained as the image under f of the "unphysical" period 2π. This would yield a "generalized third Kepler law."

Essentially the same argument works for the tightly bound motions. The restriction of f to E_- is injective, but not onto: The inverse (4.4) (with the lower sign now) maps the $\Xi^*\times\mathbb{R}=\{^-R>|s|,\ P=0,\ T\in\mathbb{R}\}$ into $S\times\mathbb{R}$, again in a many-to-one manner. The space of regularized negative-energy motions is thus once more the orbit M_s and therefore carries a symplectic action of the conformal group $C^{\uparrow}_{+}(3,1)$.

Let us now consider the *MIC–Zwanziger system* (1.3) and (1.4). In the bound motion region $0<e<q^2/2$ we apply the transformation similar to the one used in the Kepler problem,[17] namely

$$\mathbf{R}=\mathbf{r}\sqrt{q^2-2h(\mathbf{r},\mathbf{p})},\quad \mathbf{P}=\mathbf{p}/\sqrt{q^2-2h(\mathbf{r},\mathbf{p})},$$
$$T=[\sqrt{q^2-2h(\mathbf{r},\mathbf{p})}/q^2]((q^2-2h(\mathbf{r},\mathbf{p}))t+\mathbf{r}\cdot\mathbf{p}). \tag{4.7}$$

The transformation (4.7) maps the MIC–Zwanziger system into the "unphysical" one and the pullback by (4.7) of the "unphysical" presymplectic form Σ_s is $\Omega\ +dh\wedge dt$, where h is the Hamiltonian (1.4). Therefore, Eqs. (4.7) are canonical. The energy transforms as

$$H=q^2/\sqrt{q^2-2h}\,. \tag{4.8}$$

Since $q\neq 0$ by assumption, (4.7) can be inverted:

$$r_1=[H(R_iP_i)/q^2]R_i,\quad P_i=[q^2/H(R_i,P_i)]P_i,$$
$$t=(H(R_i,P_i)/q^2)^2[H(R_i,P_i)T-R_iP_i]. \tag{4.9}$$

No regularization is needed in this case because the "MIC–Zwanziger" system is itself regular: No motion reaches the center. This is clear from $r=(H/|q|)(R/|q|)\geqslant R/|q|$.

The point $r=1,\ p=0$ (the image of $R=|s|,\ P=0$) is now a regular equilibrium point. It is just where $V(r)=q^2(1-1/r)^2$ takes its minimum. It has no physical role, however, because the "MIC–Zwanziger" approximation to test particle motion in a self-dual monopole field already breaks down for much larger distances.

We conclude that for the MIC–Zwanziger system, (4.7) is a *global symplectomorphism*. The interpretation of the symmetry generators is analogous: For example, **K** corresponds to the rescaled Runge–Lenz vector

$$\mathbf{k}=(1/\sqrt{q^2-2h})(\mathbf{p}\times\mathbf{j}-q^2(\mathbf{r}/r)), \tag{4.10}$$

etc. The trajectories are ellipses in the plane perpendicular to

$$\mathbf{n}=q\mathbf{k}+(q^2/\sqrt{q^2-2h})\mathbf{j}. \tag{4.11}$$

This proves the $C^{\uparrow}_{+}(3,1)$ dynamical symmetry for the MIC–Zwanziger system, with generators given in (3.11), cf. Refs. 12 and 15. As a secondary result, we also obtain the equivalence between the regularized Taub–NUT and MIC–Zwanziger systems, cf. Ref. 25.

V. UNBOUND MOTIONS

Now we give a brief account of the unbound motions. We start with another "unphysical" system described by $\mathscr{M}_s=M_s\times\mathbb{R}$ and $\sigma_s=\varpi_s+dH\wedge dT$ [thus far identical to (3.15)], but instead choose the Hamiltonian

$$H=\Gamma_4=\tfrac{1}{2}\{R(\mathbf{P}^2-1)+s^2/R\}. \tag{5.1}$$

All motions of this system are infinite and thus the space of motions $\mathscr{M}_s/\mathrm{Ker}\,\sigma_s$ is globally symplectomorphic to the $T=0$ phase space, which is again (M_s,ϖ_s).

The generators (3.11) of the action of the conformal group are extended into $\mathscr{M}_s$ as

$$\Gamma_{4}^{\sim}=\Gamma_4, \tag{5.2a}$$
$$\mathbf{J}^{\sim}=\mathbf{J}, \tag{5.2b}$$
$$\mathbf{U}^{\sim}=\mathbf{U}, \tag{5.2c}$$
$$\mathbf{K}^{\sim}=\mathbf{K}\,\mathrm{ch}\,T+\mathbf{V}\,\mathrm{sh}\,T, \tag{5.2d}$$
$$D^{\sim}=D\,\mathrm{ch}\,T+\Gamma_0\,\mathrm{sh}\,T, \tag{5.2e}$$
$$\mathbf{V}^{\sim}=\mathbf{K}\,\mathrm{sh}\,T+\mathbf{V}\,\mathrm{ch}\,T, \tag{5.2f}$$
$$\Gamma_0^{\sim}=D\,\mathrm{sh}\,T+\Gamma_0\,\mathrm{ch}\,T. \tag{5.2g}$$

Combing with (3.12) we deduce that the trajectories are

$$\mathbf{R}(T)=\mathbf{U}(T)-\mathbf{K}(T)$$
$$=\mathbf{U}-\mathbf{K}^{\sim}\,\mathrm{ch}\,T+\mathbf{V}^{\sim}\,\mathrm{sh}\,T, \tag{5.3}$$

which are hyperbolas with the center at **U** and perpendicular to

$$\mathbf{N}=\mathbf{K}^{\sim}\times\mathbf{V}^{\sim}=s\mathbf{K}+\Gamma_4\mathbf{J}. \tag{5.4}$$

For the initial condition $\mathbf{R}(0)=\mathbf{R}_0$, $(0<|\mathbf{R}_0|<\infty)$, $\mathbf{P}(0)=0$, we obtain a semi-infinite radial motion

$$\mathbf{R}(T)=(\mathbf{R}_0/R_0)(\Gamma_0^{\sim}\mathrm{ch}\,T-\Gamma_4^{\sim}),$$
$$\mathbf{P}(T)=\left(\frac{\mathbf{R}_0}{R}\right)\frac{1}{1-(\Gamma_4^{\sim}/\Gamma_0^{\sim})(1/\mathrm{ch}\,T)}\,\mathrm{th}\,T, \tag{5.5}$$

whose (unique) turning point is at $R(0)=R_0$. The set of turning points is

$$\Delta\times\mathbb{R}=\{(\mathbf{R},\mathbf{P},T)\,|\,0<R<\infty,\mathbf{P}=\mathbf{0}\}. \tag{5.6}$$

Now we turn to the "physical" systems. Let us first assume that we are working with the $m<0$ Taub–NUT case and with the energy $e>(q/4m)^2/2$, so that the motions are hyperbolas. As for the bound case, nonradial motions avoid the singular sphere S. All radial motions hit S in finite time, with infinite velocity and zero momentum. Such a motion should be identified with the one bouncing off at the same moment along the same phase-space trajectory. An unbound motion has a single turning point, which lies in $S\times\mathbb{R}$.

Let us now relate these two systems by an appropriately modified version of (4.1), $f(\mathbf{r},\mathbf{p},t)=(\mathbf{R},\mathbf{P},T)$, with

$$\mathbf{R}=\sqrt{2h-(q/4m)^2}\,\mathbf{r},\quad \mathbf{P}=\mathbf{p}/\sqrt{2h-(q/4m)^2},$$
$$T=\frac{\sqrt{2h-(q/4m)^2}}{-4mh}\,(\mathbf{p}\cdot\mathbf{r}-(2h-(q/4m))^2t). \tag{5.7}$$

Again, (5.7) is canonical and $f^*(\varpi_s+dH\wedge dT)=\sigma$ for

$$H=4m[h-(q/4m)^2]/\sqrt{2h-(q/4m)^2}=\Gamma_4. \tag{5.8}$$

The same argument as for the bound motions shows that f is injective, but not surjective: $\mathrm{Im}(f)=\mathscr{M}_s\setminus(\Delta\times\mathbb{R})$. The formal inverse of (5.7) carries $\Delta\times\mathbb{R}$ into $\{r=4|m|,\ p=0\}\times\mathbb{R}$. In this case, the regularization amounts to restoring the turning-point set $\Delta\times\mathbb{R}$. The space of regularized, hyperbolic Taub–NUT motions hence becomes *globally symplectomorphic* to (M_s,ϖ_s). Therefore, it carries an action of the conformal *group* $C^{\uparrow}_{+}(3,1)$. This is in contrast with what occurs for the Kepler problem, where the scattered motions only have a Lie algebra symmetry, which does

TABLE I. Regularization and group action in various cases.

		Bound	Unbound		
		Regularization	Group action	Regularization	Group action
Taub–NUT	$m>0$	no bound motions		no	yes
	$m<0$	yes	yes	yes	yes
Asymptotic BPS		no	yes	no	no

not integrate into a group action.[18] Those generators commuting with the Hamiltonian (5.1), namely **J** and **U**, form an o(3,1) subalgebra. It is now **U** (rather than **K**) that goes into the rescaled Runge–Lenz vector **k** under f. This is not a surprise since (5.7) could have been obtained by requiring (besides canonicity) that the time-independent o(3,1) algebras go into each other.

The remaining cases are analogous: For $m>0$, the original Kaluza–Klein monopole situation, the metric is everywhere regular, including at the origin.[26] All motions are hyperbolic and none reaches the center, but rather has a turning point [still given by (2.11)]. The transformation (5.7) yields a *global symplectomorphism* between the phase space N_0 (which is now a global chart of the space of motions for $q\neq 0$) and M_s. Therefore, we have a global $C^{\uparrow}_{+}$ (3,1) conformal symmetry.

For a test particle in the long-range self-dual background unbound motions arise for $e \gg q^2/2$. No motion reaches the center and thus no regularization is necessary. Equation (4.12) (with a sign change under the root) is again an injective symplectic mapping: Its image is, however, only the *positive-energy part* $H>0$ of $\mathscr{M}_s$. Therefore, there is only a Lie *algebra* action, which does not integrate into a *group* action because the group trajectories leave the positive-energy part. The situation is summarized in Table I.

VI. OTHER o(2,4)-SYMMETRIC GEODESICS

Curiously enough, the same type of o(2,4) symmetry is encountered for the geodesics of some other metrics. Let us first consider the metric obtained from the Taub–NUT line element (1.1) by "rescaling":

$$ds^2 = \{dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\} + \frac{(d\psi + \cos\theta\, d\phi)^2}{(1-1/r)^2}. \tag{6.1}$$

Here $\partial/\partial\psi$ is a Killing vector, the Kaluza–Klein analog of an internal symmetry. The associated conserved quantity $q = (1-1/r)^2(\dot\psi + \cos\theta\dot\phi)$ is again an electric charge. The geodesics of (6.1) satisfy

$$\frac{d^2 r_i}{dt^2} = \alpha\frac{r_i}{r^3} + q\epsilon_{ijk}\frac{r_j v_k}{r^3} + q^2\frac{r_i}{r^4}, \tag{6.2}$$

where $\mathbf{v} = \dot{\mathbf{r}}$ and $\alpha = q^2/4m$. However, this is exactly the equation of motion one obtains for a test particle in the asymptotic field of a self-dual monopole whose electric charge is q.[5] Observe that (6.2) is the equation of motion for the MIC–Zwanziger system (1.3) and (1.4) (with the Coulomb coefficient replaced by α); thus it admits a $C^{\uparrow}_{+}$ (3,1) conformal dynamical symmetry with all its aforementioned consequences.

The metric (6.1) has the Kaluza–Klein form

$$g_{\mu\nu} = \begin{pmatrix} g_{ij} + A_i A_j/V & A_i/V \\ A_j/V & 1/V \end{pmatrix}, \tag{6.3}$$

where $g_{ij} = \delta_{ij}\,(i,j=1,2,3)$ is the flat Euclidean metric, A_i is a Dirac monopole vector potential, and the "Brans–Dicke" scalar $V = (1-1/r)^2$ is the square of the asymptotic Higgs field of a BPS monopole: It can therefore be considered as a *curved-space model* for a test particle in *the long-range self-dual monopole field*. The metric (6.1) is singular at $r=1$, yielding a singularity in the definition of the electric charge q. This is consistent with the behavior of a test particle in the monopole field. (Both the "MIC–Zwanziger" approximation and the definition of the electric charge are only valid for $r \gg 1$.)

Yet another example was found very recently by Gibbons and Ruback,[27] who consider a closed string (a "winder") in a five-dimensional static Kaluza–Klein space-time g_{AB} $[A,B = 0,1,2,3,5]$,

$$g_{AB} = \begin{pmatrix} -g_{00} & & \\ & g_{ij} + A_i A_j/V & A_i/V \\ & A_j/V & 1/V \end{pmatrix} \tag{6.4}$$

$(g_{00}=1)$. The string motion is governed by the Nambu–Goto action

$$S = -\frac{1}{2\pi\alpha'}\int\sqrt{-\det g_{AB}\frac{\partial x^A}{\partial u^a}\frac{\partial x^B}{\partial u^b}}\, du^1\, du^2, \tag{6.5}$$

where $u^1 = \sigma$ is periodic with period 2π since the string is closed and $u^2 = \tau$. Gibbons and Ruback[27] assume that the string moves entirely in the internal space, winding m times around the internal circle: More precisely, they assume that $x^5 = mR_K\sigma$, $x^\alpha = x^\alpha(\tau)$, where R_K is the radius of the internal circle at infinity. Substituting this ansatz into (6.5) and integrating over s reduces the Nambu–Goto action into that of a relativistic particle with rest mass $m = mR_K/\alpha'$:

$$S = -\left(\frac{mR_K}{\alpha'}\right)\int\sqrt{-h_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}\, d\tau, \tag{6.6}$$

where the new metric $h_{\mu\nu}$ is $h_{\mu\nu} = g_{\mu\nu}/V$. If, in particular, the original K–K metric is that of a Kaluza–Klein monopole $V = 1 + R_k/2r$, the new metric is simply

$$ds^2 = -\frac{dt^2}{1+R_k/2r} + d\mathbf{r}^2. \tag{6.7}$$

The geodesics correspond to the Hamiltonian

$h=\sqrt{(\mathbf{p}^2+\mu^2)/V}$ and phase space symplectic structure $d\mathbf{r}\wedge d\mathbf{p}$, where $\mathbf{p}=\mu\, d\mathbf{x}/dt$. Gibbons and Ruback[27] then point out that the geodesics of (6.7) lie in the plane perpendicular to the conserved angular momentum $\mathbf{j}=\mathbf{r}\times\mathbf{p}$ and are ellipses, parabolas, or hyperbolas depending on the energy *square* e^2 being smaller, equal, or larger than the rest-mass square μ^2. This is explained by the conservation of a "Runge–Lenz" vector

$$\mathbf{a}=\mathbf{p}\times\mathbf{j}-(\mathbf{r}/r)(e^2R_k/4). \tag{6.8}$$

Furthermore, $\mathbf{j}$ and $\mathbf{k}=\mathbf{a}/\sqrt{(\mu^2-h^2)}$ generate a *Kepler*-type (in contrast to the "MIC–Zwanziger-type") o(4)/o(3,1) dynamical symmetry. The energy levels

$$e^2=\mu^2(8n^2/\mu^2R_K^2)\,(\sqrt{1+\mu^2R_K^2/4n^2}-1) \tag{6.9}$$

($n=1,2,\ldots$) are n^2 degenerate.

Our method provides an insight into the above statements. One inverts the energy relation

$$n=(R_k/4)(e^2/\sqrt{\mu^2-e^2}). \tag{6.10}$$

Define now a transformation $(\mathbf{R},\mathbf{P},T)=f(\mathbf{r},\mathbf{p},t)$:

$$\mathbf{R}=\sqrt{\mu^2-h^2}\mathbf{r},\quad \mathbf{P}=\mathbf{p}/\sqrt{\mu^2-\eta^2},$$

$$T=\frac{4}{R_k}\frac{\sqrt{\mu^2-h^2}}{2\mu^2-h^2}\left[\frac{\mu^2-h^2}{h}t+\mathbf{r}\cdot\mathbf{p}\right]. \tag{6.11}$$

It is easy to see that (6.11) is canonical, $d\mathbf{R}\wedge d\mathbf{P}+dH\wedge dT=d\mathbf{r}\wedge d\mathbf{p}+dh\wedge dt$, if the new Hamiltonian is

$$H=(R_k/4)(h^2/\sqrt{\mu^2-h^2}). \tag{6.12}$$

Substituting h, expressed by $\mathbf{R}$ and $\mathbf{P}$, into (6.12), we obtain

$$H=\Gamma_0=\tfrac{1}{2}R(\mathbf{P}^2+1), \tag{6.13}$$

which is the SU(2,2) generator (3.11a) for helicity $s=0$, i.e., the Hamiltonian of the geodesic flow on S^3 expressed in stereographic coordinates.

We conclude that f in (6.11) is an (injective) symplectic mapping from the "reduced string system" into the mass-zero helicity-zero SU(2,2) orbit $\mathcal{O}_0\simeq T^+S^3$, which is[16,18] the space of regularized motions of the *Kepler problem*. In this case $\mathbf{R}$ and $\mathbf{P}$ are only local coordinates obtained by stereographic projection. Now f is not onto; those points not in Im(f) can be used to regularize the problem along the same lines as before.

It follows that the geodesics of the metric (6.7) have an o(4,2) conformal symmetry, with the generators (3.11) (for $s=0$).

VII. CONCLUSION

In this paper we have only studied the classical mechanics. Quantum aspects are found in Refs. 1–4 and could (in principle) be obtained from implementing the canonical transformation (4.1) at the quantum level.

The complications arise because of the collisions, which require regularization. The quantum motions actually behave better than the classical ones: Intuitively, the Heisenberg uncertainty relations make the collisions irrelevant. Remarkably, it is for the radial motions that the "Atiyah–Hitchin" and "Taub–NUT" motions differ the most.[1]

The Taub–NUT approximation is only valid for large distances, when the exponential terms are small with respect to those in r^{-1}. In the "real" (Atiyah–Hitchin) case the relative electric charge may not be conserved; the trajectories may not stay in a plane, etc.[1,28] However, numerical as well as theoretical calculations[28] show that the system still admits bound motions; for large angular momentum the real spectrum is very close to the one in the Taub–NUT limit.

An isospinor test particle in the long-range field of a monopole[5,11] has similar properties. In particular, for large angular momentum, the "real" bound motions peak far outside the monopole core and the spectrum quickly converges to the "MIC–Zwanziger" one.[29]

Finally, notice that the evolution space formalism has been useful in the past in understanding the symmetries of the Dirac monopole.[30]

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