ANNALS OF PHYSICS 203, 76–136 (1990)

Toda Theory and $\mathcal{W}$-Algebra from a Gauged WZNW Point of View

J. BALOG,* L. FEHÉR,† AND L. O'RAIFEARTAIGH

Dublin Institute for Advanced Studies, 10 Burlington Road, Dublin 4, Ireland

P. FORGÁCS*

Max-Planck-Institut für Physik und Astrophysik (Werner-Heisenberg-Institut für Physik), P.O. Box 401212, Munich, Federal Republic of Germany

AND

A. WIPF

Institüt für Theoretische Physik, ETH-Hönggerberg, CH-8093 Zürich, Switzerland

Received December 14, 1989

A new formulation of Toda theories is proposed by showing that they can be regarded as certain gauged Wess–Zumino–Novikov–Witten (WZNW) models. It is argued that the WZNW variables are the proper ones for Toda theory, since all the physically permitted Toda solutions are regular when expressed in these variables. A detailed study of classical Toda theories and their $\mathcal{W}$-algebras is carried out from this unified WZNW point of view. We construct a primary field basis for the $\mathcal{W}$-algebra for any group, we obtain a new method for calculating the $\mathcal{W}$-algebra and its action on the Toda fields by constructing its Kac–Moody implementation, and we analyse the relationship between $\mathcal{W}$-algebras and Casimir algebras. The $\mathcal{W}$-algebra of G_2 and the Casimir algebras for the classical groups are exhibited explicitly.

I. INTRODUCTION

Two dimensional conformally invariant soluble field theories are based on various extensions of the chiral Virasoro algebras. The best known extension is the Kac–Moody (KM) extension [1], whose most prominent Lagrangean realization is the Wess–Zumino–Novikov–Witten (WZNW) model [2]. There are various indications that the KM algebra may even underlie all the rational conformal field theories. For example, the Goddard–Kent–Olive (GKO) construction [3]

* On leave from Central Research Institute for Physics, Budapest, Hungary.

† On leave from Bolyai Institute, Szeged, Hungary.

0003-4916/90 $7.50

generates a huge class of rational conformal field theories. Another extension is the so-called $\mathcal{W}$-extension, which is a polynomial extension of the Virasoro algebra by higher spin fields. The study of polynomial extensions of the Virasoro algebra was initiated by Zamolodchikov [4]. Later it was realized [5, 6] that a large class of polynomial extensions of the Virasoro algebra can be constructed by quantizing the second Gelfand–Dickey Poisson bracket structure of Lax operators, used in the theory of integrable systems. These $\mathcal{W}$-algebras proved very fruitful in analysing conformal field theories and they have become the subject of intense study [5–8]. Recently it has been found by Gervais and Bilal that Toda theories provide a realization of $\mathcal{W}$-algebras [8, 9]. Toda theories are important in the theory of integrable systems and include the ubiquitous Liouville theory, which, among other things, describes two dimensional induced gravity in the conformal gauge.

There are a number of results suggesting that Toda theories must be closely related to WZNW models. First, in both cases the fields can be recovered from the generators of the respective extended Virasoro algebras (KM and $\mathcal{W}$-algebras) by means of linear differential equations [8]. Second, the Gelfand–Dickey Poisson bracket structure can be obtained by a Hamiltonian reduction from a KM phase space [10]. Finally, it has been shown by Polyakov [11] that two dimensional induced gravity (in the light cone gauge) exhibits (left-moving) $SL(2, R)$ KM symmetry.

In a recent letter [12] we have shown that the exact relationship is that Toda theories may be regarded as WZNW models (based on maximally non-compact, simple real Lie groups) reduced by certain conformally invariant constraints. To be more precise, Toda theory can be identified as the constrained WZNW model, modulo the left-moving upper triangular and right-moving lower triangular KM transformations, which are gauge transformations generated by the constraints. The advantages of treating Toda theory as a gauge theory embedded into a WZNW model are the following: First, the coordinate singularities of Toda theory disappear by using the WZNW variables. Second, the $\mathcal{W}$-algebra of Toda theory arises immediately as the algebra formed by the gauge invariant polynomials of the constrained KM currents and their derivatives. Third, the general solution of the Toda field equations is easily obtained from the very simple WZNW solution. Finally, there are natural gauges which facilitate the analysis of the theory. In this paper we exploit the embedding of Toda theory into the WZNW model to obtain a number of new insights and results about the structure of Toda theory and $\mathcal{W}$-algebra. All our considerations are classical. We hope that their quantum generalizations will provide new constructions of quantum Toda theories [13] and $\mathcal{W}$-algebras.

We first set up a Lagrangean framework for the WZNW-Toda reduction, namely we establish that Toda theories can be identified as the gauge invariant content of certain gauged WZNW models. Our gauged WZNW models differ from the usual gauged WZNW models [14] used in the path integral realization of the GKO construction not only in the non-compactness of our groups, but also in that instead of a single diagonal subgroup we gauge two subgroups of the left $\times$ right WZNW group, the upper triangular maximal nilpotent subgroup on the left and the lower triangular one on the right hand side. The nilpotency of the triangular subgroups is crucial to this ambidextrous generalization of the usual vector gauged WZNW models, and in fact the nilpotency of the gauge group is the reason for the appearance of the simple polynomial structures in Toda theory. The constrained WZNW model is recovered in this framework by an appropriate partial gauge fixing which leaves the left- and right-moving triangular gauge transformations mentioned earlier as a residual gauge symmetry.

In most of our considerations we rely heavily on the use of a class of natural gauges used in studying the gauge invariant differential polynomials in the review paper [10] by Drinfeld and Sokolov. The basic property which makes the DS gauges convenient is that in each DS gauge the surviving components of the KM current serve as a basis for the $\mathcal{W}$-algebra.

Working in the DS gauges, we give a simple algorithm to find the KM transformations which implement the canonical transformations generated by the $\mathcal{W}$-algebra. This provides us with a new method both for computing the $\mathcal{W}$-algebra and for determining the action of the $\mathcal{W}$-algebra on the Toda fields. Our method crucially depends on using the embedding WZNW theory and its full, non-constrained KM algebra. We illustrate the method on the examples of A_2 and B_2 and demonstrate its power by computing the complete $\mathcal{W}$-algebra relations for the rather non-trivial example of G_2.

We find a DS gauge which enables us to construct a primary field basis of the $\mathcal{W}$-algebra. As far as we know a general algorithm for constructing primary $\mathcal{W}$-generators has not been known before, although such generators have been found in low dimensional examples [6]. We note that even the existence of a primary field basis is not completely trivial, since such a basis is constructed by a non-linear transformation [6] even if one starts from $\mathcal{W}$-generators transforming in a linear (inhomogeneous) manner under the Virasoro algebra. Our construction of the primary $\mathcal{W}$-generators is based on a special $SL(2, R)$ subgroup of the WZNW group, which plays an important role throughout the theory. The primary $\mathcal{W}$-generators are associated in a natural way to the highest weight states of this $SL(2, R)$ in the adjoint representation of the WZNW group.

There have been attempts [15] at constructing polynomial extensions of the Virasoro algebra from a KM algebra by using the higher Casimirs of the underlying Lie algebra similar to the manner in which the second order Casimir is used in the Sugawara construction. On the quantum level these Casimir algebras close only under very restrictive conditions on the KM representation. We show that the leading terms (i.e., terms without derivatives) of the $\mathcal{W}$-generators are always Casimirs, and that the Poisson bracket version of the Casimir algebras always close. In fact, we prove that these classical Casimir algebras are obtained from the corresponding $\mathcal{W}$-algebras by a certain truncation, and thus the Casimir algebras can be used to investigate the leading terms of the $\mathcal{W}$-algebras. For the classical Lie algebras A_l, B_l, and C_l we give the explicit form of the Casimir algebra.

We also consider the existence of quadratic relations for the $\mathcal{W}$-algebras. In the case of the A_l, B_l, and C_l Lie algebras it is easy to display $\mathcal{W}$-generators with quadratic relations. The above mentioned relation between the Casimir and $\mathcal{W}$-algebras shows that for the other Lie algebras the $\mathcal{W}$-relations are necessarily of higher order.

Finally, we investigate how the Toda fields can be reconstructed from the $\mathcal{W}$-generators. This reconstruction is a reduced version of the reconstruction of the group valued WZNW field from the KM currents, and this tells us that every Toda solution with regular $\mathcal{W}$-generators can be represented by a regular WZNW solution. The reconstruction problem leads us to studying the differential equations satisfied by the gauge invariant components of the constrained WZNW field. This way we recover the Lax operators studied in [10], which also appear in the generalized Schrödinger equations of Ref. [8]. For A_l, B_l, C_l, and G_2 the reconstruction problem can be reduced to solving a single ordinary differential equation of the order of the defining representation of the corresponding algebra, in all other cases one inevitably has a pseudo-differential equation. We will see that one has a single ordinary differential equation exactly when the representation in which the group valued WZNW field is taken is irreducible under the $SL(2, R)$ subgroup mentioned earlier, and that in general the structure of the pseudo-differential operator depends on the decomposition of this representation under the $SL(2, R)$ subgroup.

The plan of the paper is the following: In Section II we present a short review of the reduction of the WZNW model to Toda theory and describe the gauged WZNW framework. We elaborate on the role of the residual gauge invariance and on the gauge invariant quantities in Subsection II.2. The longest and most important section is III. We start it with the definition of the $\mathcal{W}$-algebras. In Subsection III.1 we present the construction of the Drinfeld–Sokolov gauges and observe that in these gauges the $\mathcal{W}$-algebra reduces to the Dirac bracket algebra of the surviving KM current components. In Subsection III.2 we exhibit a primary field basis of the $\mathcal{W}$-algebra and illusrate it with B_2. In Subsection III.3 we give an algorithm to implement the action of the $\mathcal{W}$-algebra by means of KM transformations and illustrate the procedure with A_2 and B_2. In Subsection III.4 we first display a subclass of Drinfeld–Sokolov gauges where the $\mathcal{W}$-algebra relations are quadratic for A_l, B_l, and C_l. Then we introduce the "diagonal" gauge, which is frequently used in Section IV, and briefly discuss the related Miura-transformation. Subsection IV.1 contains a detailed analysis of the relation between the Casimir Poisson bracket algebras and the $\mathcal{W}$-algebras. In Subsection IV.2 we present the explicit Poisson bracket algebra of the Casimir operators of the classical Lie algebras A_l, B_l, and C_l. In the last section, V, we study the differential and pseudo-differential operators which appear when the Toda-fields are reconstructed from the $\mathcal{W}$-generators (or the constrained WZNW fields are reconstructed from the KM currents). There are three appendixes; Appendix A contains our conventions and some important group theoretical results, Appendix B contains the complete $\mathcal{W}(G_2)$ algebra and Appendix C contains the details of the calculations of the Casimir algebras. We end the paper by summarizing the main results and giving some conclusions.

II. Toda Field Theory as a Gauge Theory

In this section we first summarize the main points of the redution of WZNW models to Toda theories. Then we show how to set up a Lagrangean framework for the reduction, using an ambidextrous generalization of the usual vector gauged WZNW models. Then we elaborate on the concept of residual gauge transformations and on the corresponding gauge invariant quantities. In particular, we point out that in the WZNW framework $\mathcal{W}$-algebras appear naturally as symmetry algebras of Toda theory.

II.1. *Toda Theory as a Gauged WZNW Model*

The so called Toda field equations constitute a rather interesting set of integrable (soluble) equations. These equations appear naturally in various problems (cylindrically symmetric instantons [16], etc.) and they can also be thought of as a generalization of the ubiquitous Liouville equation:

$$\partial_+ \partial_- \phi + M e^{\phi} = 0, \qquad \text{where} \quad M = \text{const.}. \tag{2.1}$$

Now the Toda equations are given as

$$\partial_+ \partial_- \phi^\alpha + \frac{1}{2} |\alpha|^2 M^\alpha \exp\left\{\frac{1}{2} \sum_{\beta \in \Delta} K_{\alpha\beta} \phi^\beta\right\} = 0, \tag{2.2}$$

where $K_{\alpha\beta}$ is the Cartan matrix[1] of a simple Lie algebra, Δ denotes the set of simple roots and the M^α's are (positive) constants. The corresponding Lagrangean is

$$\mathscr{L} = \frac{\kappa}{2}\left(\sum_{\alpha, \beta \in \Delta} \frac{1}{2\, |\alpha|^2} K_{\alpha\beta}\, \partial_+ \phi^\alpha\, \partial_- \phi^\beta - \sum_{\alpha \in \Delta} M^\alpha \exp\left\{\frac{1}{2} \sum_{\beta \in \Delta} K_{\alpha\beta} \phi^\beta\right\}\right), \tag{2.3}$$

where κ is the coupling constant of the theory. Clearly (2.2) reduces to the Liouville equation (2.1) by making the simplest choice for $K_{\alpha\beta}$, namely the choice when $K_{\alpha\beta}$ is just a number (corresponding to a rank one algebra). In fact Toda field theories are also distinguished by being the only two dimensional, nontrivial conformally invariant models which are soluble [8, 16] in the class of scalar theories without derivative couplings.

These theories possess an imporved energy–momentum tensor

$$\Theta_{\pm\pm} = \frac{\kappa}{2}\left(\sum_{\alpha, \beta \in \Delta} \frac{1}{|\alpha|^2} K_{\alpha\beta}\, \partial_\pm \phi^\alpha\, \partial_\pm \phi^\beta - 4 \sum_{\alpha \in \Delta} \frac{1}{|\alpha|^2} \partial_\pm^2 \phi^\alpha \right) \tag{2.4}$$

with vanishing trace, $\Theta_{+-} = 0$, on shell. Interestingly, the general solution of (2.2) can be written in closed form [16].

[1] Our conventions are collected in Appendix A.

Let us recall first, how Toda theories can be regarded as constrained WZNW models. We start with the WZNW action based on a connected real Lie group G (with maximally non-compact simple real Lie algebra $\mathcal{G}$)

$$S(g) = -\frac{k}{8\pi}\int d^2x\eta^{\mu\nu}\,\mathrm{Tr}\{(g^{-1}\partial_\mu g)(g^{-1}\partial_\nu g)\} + \frac{k}{12\pi}\int_{B_3}\mathrm{Tr}\{(g^{-1}dg)^3\}, \tag{2.5}$$

where g is a group-valued field and B_3 is a three dimensional manifold whose boundary is Minkowski space–time. We choose the coupling constants κ and k to be related by the equality $k = -4\pi\kappa$.

This action possesses left and right KM symmetries. Their Noether currents associated to some Lie algebra element, λ, are given as

$$\begin{aligned} J(\lambda) &= \mathrm{Tr}(\lambda\cdot J), \qquad J = \kappa(\partial_+ g)g^{-1} \\ \tilde{J}(\lambda) &= \mathrm{Tr}(\lambda\cdot\tilde{J}), \qquad \tilde{J} = -\kappa g^{-1}(\partial_- g). \end{aligned} \tag{2.6}$$

The field equations are equivalent to the conservation of the left and right currents:

$$\partial_- J = 0, \qquad \partial_+\tilde{J} = 0. \tag{2.7}$$

Let now μ^α and ν^α ($\alpha \in \Delta$) be arbitrary positive numbers and let us denote the set of positive roots by Φ^+. The main result of Ref. [12] was that by imposing the constraints,

$$\begin{aligned} J(E_\alpha) &= \kappa\mu^\alpha, \qquad \tilde{J}(E_{-\alpha}) = -\kappa\nu^\alpha, \qquad \alpha\in\Delta \\ J(E_\varphi) &= 0, \qquad \tilde{J}(E_{-\varphi}) = 0, \qquad \varphi\in\Phi^+\backslash\Delta \end{aligned} \tag{2.8}$$

the equations of motion of the WZNW theory (2.7) reduce to the Toda field equations (2.2). To prove this result we start with the (local) Gauss decomposition

$$g = ABC \tag{2.9a}$$

of the group-valued field g, where

$$\begin{aligned} A &= \exp\left\{\sum_{\varphi\in\Phi^+} a^\varphi E_\varphi\right\}, \qquad C = \exp\left\{\sum_{\alpha\in\Phi^+} c^\varphi E_{-\varphi}\right\}, \\ B &= \exp\left\{\frac{1}{2}\sum_{\alpha\in\Delta}\phi^\alpha H_\alpha\right\}. \end{aligned} \tag{2.9b}$$

This group-valued Gauss decomposition is locally unique for Lie groups G with maximally non-compact Lie algebras.

Now exploiting the fact that μ^α and ν^α are zero for all but the simple roots, the constraints (2.8) can be rewritten as

$$A^{-1}\partial_- A = \sum_{\alpha\in\Delta} \frac{1}{2}|\alpha|^2 \nu^\alpha E_\alpha \exp\left\{\frac{1}{2}\sum_{\beta\in\Delta} K_{\alpha\beta}\phi^\beta\right\}$$
$$(\partial_+ C)C^{-1} = \sum_{\alpha\in\Delta} \frac{1}{2}|\alpha|^2 \mu^\alpha E_{-\alpha} \exp\left\{\frac{1}{2}\sum_{\beta\in\Delta} K_{\alpha\beta}\phi^\beta\right\}. \tag{2.10}$$

Substituting (2.10) into the field equations (2.7) one indeed recovers the Toda equations (2.2) (with $M^\alpha = |\alpha|^2 \mu^\alpha \nu^\alpha$). It can be shown that this reduction is canonical in the sense that the Poisson brackets of the Toda variables ϕ and $\dot{\phi}$ can be calculated either from the Toda or from the WZNW action (as a requirement, this fixes the relationship between the coupling constants).

We remark that the famous Leznov–Savaliev general solution of the Toda field equations [16] can be derived effortlessly from the general solution of the WZNW field equations (2.7),

$$g(x^+, x^-) = g_L(x^+)\cdot g_R(x^-), \tag{2.11}$$

where g_L and g_R are arbitrary group-valued functions constrained only by the boundary conditions and there is an obvious constant-matrix ambiguity in the definition of g_L and g_R. The general solution of the Toda equations can be obtained from (2.11) by first imposing the constraints (2.8) and then decomposing the constrained WZNW solution according to (2.9) [12]. In Section V we shall show that it is equally easy to recover the solution of the Toda field equations in the form recently found by Gervais and Bilal [8] from (2.11).

As the quantization of Liouville and Toda theories is expected to be simpler in the WZNW formulation, it is worthwhile to find a Lagrangean realization of the reduction of the WZNW model to the Toda theory. In the following we show that an ambidextrous generalization of gauged WZNW models [14] provides a natural framework to carry out this reduction. For example, gauged WZNW models turned out to be useful in the Lagrangean description of the Goddard–Kent–Olive coset construction (GKO) [3].

We shall need the Polyakov–Wiegmann identity [17] expressing the WZNW action for the product of three matrices A, B, C as the sum of the respective actions for A, B, and C, modulo local terms:

$$S(ABC) = S(A) + S(B) + S(C)$$
$$+\kappa\int d^2x\,\mathrm{Tr}\{(A^{-1}\partial_- A)(\partial_+ B)B^{-1}$$
$$+(B^{-1}\partial_- B)(\partial_+ C)C^{-1} + (A^{-1}\partial_- A)B(\partial_+ C)C^{-1}B^{-1}\}. \tag{2.12}$$

Next we want to consider the gauged version of the WZNW theory, i.e., we are looking for an action invariant under the transformations,

$$g \to \alpha g \beta^{-1}, \qquad \alpha \in H, \beta \in \tilde{H}, \tag{2.13}$$

where α, β are functions of both x^+ and x^-, and H, $\tilde{H}$ are two isomorphic subgroups of G.

Let us first recall the "usual" gauged WZNW models [14]. In the standard case one gauges a diagonal (vector) subgroup, H, of the Kac–Moody group $G_{\mathrm{L}} \times G_{\mathrm{R}}$. Now the transformation of g under the vector subgroup is given as

$$g \to \gamma g \gamma^{-1}, \qquad \gamma(x^+, x^-) \in H. \tag{2.14a}$$

It is easy to see that the action functional

$$I(g, h, \tilde{h}) = S(hg\tilde{h}^{-1}) - S(h\tilde{h}^{-1}), \qquad h, \tilde{h} \in H$$

is gauge invariant, provided (2.14a) is supplemented with

$$h \to h\gamma^{-1}, \qquad \tilde{h} \to \tilde{h}\gamma^{-1}. \tag{2.14b}$$

Using (2.12), $I(g, h, \tilde{h})$ can be rewritten as

$$I(g, A_-, A_+) = S(g) + \kappa \int d^2x \,\mathrm{Tr}\{A_-(\partial_+ g)g^{-1} + (g^{-1}\partial_- g)A_+ + A_- g A_+ g^{-1} - A_- A_+\}, \tag{2.15}$$

where $S(g)$ is the WZNW action (2.5) and

$$A_- = h^{-1}\partial_- h, \qquad A_+ = (\partial_+ \tilde{h}^{-1})\tilde{h}. \tag{2.16}$$

In the action functional (2.15) A_-, A_+ are regarded as the light-cone components of some "gauge field" belonging to the adjoint representation of H, transforming according to (2.16), and its gauge invariance is obvious from the above construction. The variation of this action with respect to the non-propagating gauge fields $A_\pm$ provides constraints which classically set the currents of H to zero. It has been demonstrated [14] that a careful quantization of (2.15) yields the GKO coset construction.

At first sight it seems impossible to generalize (2.15) to be invariant under the more general transformations (2.13), since now the only obvious candidate for an invariant action is just $S(hg\tilde{h}^{-1})$ which is non-local in the gauge fields. However, in the rather degenerate case when H and $\tilde{H}$ are the subgroups of G generated by the step operators associated to the positive and negative roots, and denoted by N

and $\tilde{N}$, respectively, their Lie algebras are nilpotent, and hence one has the crucial property that

$$S(h) = S(\tilde{h}) \equiv 0. \tag{2.17}$$

So $S(hg\tilde{h}^{-1}) - S(g)$ is local, therefore the gauge fields A_-, A_+ in Eq. (2.16) (where now $h \in N$ and $\tilde{h} \in \tilde{N}$) can be used in this case in the same way as for the case of a diagonal subgroup to set the corresponding N and $\tilde{N}$ currents to zero. Since the constraints we want to implement set certain currents to constants rather than to zero, we consider the action

$$I(g, A_-, A_+) = S(g) + \kappa \int d^2x \, \mathrm{Tr}\{A_-(\partial_+ g) g^{-1} + (g^{-1} \partial_- g) A_+ \\ + A_- g A_+ g^{-1} - A_- \mu - A_+ \nu\}, \tag{2.18}$$

where μ, ν are special (constant) matrices, given by

$$\nu = \sum_{\alpha \in \Delta} \frac{1}{2} |\alpha|^2 \nu^\alpha E_\alpha, \qquad \mu = \sum_{\alpha \in \Delta} \frac{1}{2} |\alpha|^2 \mu^\alpha E_{-\alpha}.$$

A_-, A_+ are now independent gauge fields in the adjoint representation of the subgroups N and $\tilde{N}$ so they are nilpotent matrices. The invariance of the action (2.18) under the gauge transformations,

$$g \to \alpha g \beta^{-1}, \qquad A_- \to \alpha A_- \alpha^{-1} + \alpha \, \partial_- \alpha^{-1}, \qquad A_+ \to \beta A_+ \beta^{-1} + (\partial_+ \beta) \beta^{-1}, \tag{2.19a}$$

where

$$\alpha = \alpha(x^+, x^-) \in N \qquad \text{and} \qquad \beta = \beta(x^+, x^-) \in \tilde{N}, \tag{2.19b}$$

is now not completely obvious because of the non-gauge-invariant looking terms, $\mathrm{Tr}(A_+ \nu + A_- \mu)$. However, these terms change by a total derivative under gauge transformations because of the special form of A_+, A_- and because the matrix ν (resp. μ) contains only step operators corresponding to simple positive (resp. negative) roots. For example, under the transformation (2.19) with

$$\beta(x^+, x^-) = \exp\left[\sum_{\varphi \in \Phi^+} \psi_\varphi E_{-\varphi} \right]$$

we have

$$\mathrm{Tr}\{\nu \cdot (\partial_+ \beta) \beta^{-1}\} = \sum_{\varphi \in \Delta} \nu^\varphi \, \partial_+ \psi_\varphi,$$

and the term Tr $A_+ v$ indeed changes only by a total derivative. The equations of motion following from (2.18) are (with $\varphi \in \Phi^+$)

$$\partial_+(g^{-1}\partial_- g + g^{-1}A_- g) - [A_+, g^{-1}\partial_- g + g^{-1}A_- g] + \partial_- A_+ = 0 \qquad (2.20a)$$

$$\partial_-(\partial_+ gg^{-1} + gA_+ g^{-1}) + [A_-, \partial_+ gg^{-1} + gA_+ g^{-1}] + \partial_+ A_- = 0 \qquad (2.20b)$$

$$\mathrm{Tr}[E_{-\varphi}(g^{-1}\partial_- g + g^{-1}A_- g - v)] = 0 \qquad (2.20c)$$

$$\mathrm{Tr}[E_\varphi(\partial_+ gg^{-1} + gA_+ g^{-1} - \mu)] = 0. \qquad (2.20d)$$

Now making use of gauge invariance, A_+ and A_- can be set equal to zero simultaneously and then we recover from (2.20) the equations of motion of the WZNW model (2.7) together with the constraints (2.8). Note, however, that setting A_+, A_- to zero is not a complete gauge fixing. Indeed, it is clear that the condition $A_\pm = 0$ is invariant under chiral gauge transformations $\alpha = \alpha(x^+)$ and $\beta = \beta(x_-)$ which are in the intersection of the gauge group and the KM symmetry group of the theory. Since in the $A_\pm = 0$ gauge (2.20) reduces to (2.7) and (2.8), it follows that the residual gauge transformations

$$g \to \alpha g \beta^{-1}, \qquad \text{where} \quad \alpha = \alpha(x^+) \in N,\ \beta = \beta(x^-) \in \tilde{N} \qquad (2.21)$$

must leave (2.8) invariant. This can also be verified by using the standard transformation property of the currents J and $\tilde{J}$ under KM transformations:

$$J \to \alpha J \alpha^{-1} + \kappa(\partial_+ \alpha)\alpha^{-1} \qquad \text{and} \qquad \tilde{J} \to \beta \tilde{J} \beta^{-1} + \kappa(\partial_- \beta)\beta^{-1}. \qquad (2.22)$$

Note that these chiral gauge transformations (2.21) form the complete residual gauge group of the gauge $A_\pm = 0$.

From now on we stay in this gauge. Here we point out how the residual gauge transformations (2.21) arise from the Hamiltonian point of view. For this, as well as in the rest of the paper, we take the space of solutions, given by (2.11), of the WZNW theory as our phase space. This is convenient here because of the left-right factorized form of the general solution. The translation to the equivalent equal time canonical formalism could be made by parametrizing the solutions by their initial data and expressing the initial data in terms of the canonical variables. To make this translation as easy as possible, in this paper we use equal time Poisson brackets on the space of solutions. After these remarks, let us observe that the KM Poisson brackets of those current components which are to be constrained according to (2.8) vanish on the submanifold of the phase space defined by (2.8) (constraint-surface), i.e., we are dealing with first class constraints. Now first class constraints always generate such canonical transformations which leave the constraint-surface invariant, and it is easy to see that in our case these are naturally identified with the residual gauge transformations.

II.2. *Gauge-Invariant Quantities*

Clearly the Toda fields, ϕ^α in (2.9b), are not affected by the residual gauge transformations (2.21). Assuming the validity of the Gauss decomposition (2.9) the Toda fields constitute a complete system of independent invariants with respect to these transformations on the "constraint-surface." In other words, Toda theory can be identified, at least locally, with the constrained WZNW model modulo residual gauge transformations. From now on we shall refer to the residual gauge transformations (2.21–2.22) simply as gauge transformations.

It is important to note, that (2.9) is valid only in a neighbourhood of the identity of G. As a consequence of this non-global nature of the Gauss decomposition, our reduction can generate singular Toda solutions from perfectly regular WZNW solutions. This is the basis of one of the most important properties of the WZNW setting of Toda theory, namely, that the physically allowed singularities of the Toda solutions are precisely those which disappear by using the WZNW variables. We have shown this in Ref. [12] in the special case of $SL(2, R)$ by proving that the requirement that a Liouville solution be obtained from a regular solution of the WZNW theory is equivalent to demanding that the associated energy–momentum tensor (2.4) be regular.

In Section V we shall show that this generalizes for a rank l algebra where besides the energy–momentum tensor there are $l-1$ additional "$\mathcal{W}$-densities." In that case the Toda solutions with regular $\mathcal{W}$-densities can be represented by regular WZNW solutions, even if they appear singular in terms of the original local Toda variables ϕ^α. It can be argued that the singular classical solutions with regular $\mathcal{W}$-densities correspond to an important sector of the quantized Toda theory. In the WZNW context these solutions are clearly on the same footing as the manifestly regular solutions. Thus the WZNW variables are the proper ones for Toda theories. However, since we must still identify gauge-related WZNW fields, we are lead to study the gauge-invariant quantities in the constrained WZNW theory.

The Toda fields ϕ^α are invariant, but they are only well defined for WZNW solutions in that neighbourhood of the identity where the Gauss-decomposition (2.9) is valid. Of course one could cover G with a finite number of patches and introduce locally regular Toda fields on them. These local fields would be related by some group transformations on the intersections of these patches and together they would define a global Toda field.

Fortunately there is a simpler and more direct way to find globally well defined quantities which reduce to the local Toda fields in the neighbourhood of the identity. Consider some (d-dimensional) representation of G and choose a basis such that the Cartan subalgebra is represented by diagonal matrices, and the Lie algebras of N and $\tilde{N}$ are represented by upper and lower triangular matrices, respectively. Then, because α and β in (2.21) are upper and lower triangular matrices, respectively, with 1's in their diagonals, it follows that the lower-right sub-determinants

$$\mathscr{D}_i \equiv \det \begin{pmatrix} g_{ii} & \cdots & g_{id} \\ \vdots & & \vdots \\ g_{di} & \cdots & g_{dd} \end{pmatrix} \tag{2.23}$$

of the matrix (g_{ij}) are all gauge-invariant quantities. It is also easy to see that in the Gauss decomposable case the Toda fields ϕ^α can be recovered as linear combinations of logarithms of the $\mathscr{D}_i$.

For example, let us consider A_l and take $G = SL(l+1, R)$ in the defining representation. Using the standard convention, in which H_{α_i} has 1 in its ii-slot, -1 in its $(i+1)(i+1)$-slot, and 0's elsewhere, for a Gauss decomposable g one obtains the simple formula,

$$\mathscr{D}_i = e^{-\phi_{l-i}/2}, \qquad \text{where} \quad \phi_k \equiv \phi^{\alpha_k}. \tag{2.24}$$

The local Toda field ϕ indeed becomes singular where the Gauss-decomposition ceases to be valid, that is where one of the sub-determinants $\mathscr{D}_i$ changes sign.

In general the globally well-defined sub-determinants (2.23) yield an overcomplete system of invariants, but in each concrete case one can single out l independent ones. For example, for the defining representations of the classical groups, the last l sub-determinants starting from g_{dd} suffice. They can be used as global variables for the Toda theory, after imposing the constraints (2.8). Since these sub-determinants are polynomial in the components of the basic WZNW field, g, they appear better suited for quantizing Toda theories than the original Toda fields themselves.

For later use we note that beside the sub-determinants, which are fully gauge-invariant polynomial quantities, there are other important quantities, which are linear in g, but invariant under left (or right) gauge transformations only. These are simply the elements of the last row (column) of g (and of g_{L} and g_{R} in (2.11), respectively).

As the KM algebra plays a central role in WZNW theories, it is clear that gauge-invariant quantities formed out of the KM current J (and $\tilde{J}$) will also be important in the Toda theories. To illustrate this, we recall how the conformal invariance of the Toda theory appears in the WZNW framework. Here we restrict ourselves to the left-moving sector. It can be shown that there is a unique Virasoro algebra in the semidirect product formed by the KM algebra and its associated Sugawara Virasoro algebra, weakly commuting with the constraints (2.8). Since the residual gauge transformations are generated by these constraints, the energy–momentum density

$$L = L^s - \mathrm{Tr}(J'\hat{\rho}), \qquad \text{where} \quad L^s = \frac{1}{2\kappa}\mathrm{Tr}(J^2), \qquad \hat{\rho} = \frac{1}{2}\sum_{\alpha \in \Phi^+} H_\alpha \tag{2.25}$$

giving rise to this Virasoro algebra, becomes gauge invariant on the constraint-surface. It follows that L must generate the conformal symmetry of the constrained WZNW, i.e., of Toda theory. (One can verify that, after imposing (2.8) and using the local coordinates defined by the Gauss decomposition (2.9), L indeed reduces to the improved energy–momentum tensor Θ_{++} (2.4).) Note that $\hat{\rho}$ in (2.25) has the property

$$[\hat{\rho}, E_\alpha] = E_\alpha, \qquad \text{when} \quad \alpha \in \Delta, \tag{2.26}$$

and that the classical centre of the (Toda) Virasoro algebra is

$$c = -12k\,\mathrm{Tr}(\hat{\rho}^2), \tag{2.27}$$

where k is the level of the underlying KM algebra.

We will see in Section III that, besides L, there are other gauge-invariant polynomial quantities formed out of the constrained KM current and its derivatives. These objects will be referred to as gauge-invariant differential polynomials.

A crucial property (which we elaborate on in Section III) of the gauge-invariant differential polynomials is that they form a closed algebra under the KM Poisson bracket operation. That is, the Poisson bracket of two gauge-invariant differential polynomials is again expressible in terms of gauge-invariant differential polynomials and δ-distributions. This means that if the quantities W^i form a basis in the ring of gauge-invariant differential polynomials then we have

$$\{W^i(x), W^j(y)\} = \sum_k P_k^{ij}(W)\,\delta^{(k)}(x^1 - y^1), \tag{2.28}$$

where the P_k^{ij} are polynomials of the W^i's and their derivatives. These Poisson bracket relations generate a non-linear algebra, reminiscent of a universal enveloping algebra.

This non-linear algebra of the gauge-invariant differential polynomials always contains the Virasoro algebra, hence it is a polynomial extension of it. This way one associates an extended conformal algebra to every Kac–Moody algebra based on maximally non-compact simple real Lie algebras, for any level k. It turns out that this polynomial algebra is always finitely generated, by $l = \mathrm{rank}(\mathscr{G})$ elements. In the literature these algebras are referred to as classical $\mathscr{W}$-algebras.

The quantum analogues of these Poisson bracket algebras play an important role in conformal field theory [4–9]. It has recently been realized [5–6] that quantum $\mathscr{W}$-algebras can be constructed by quantizing the so-called second Gelfand–Dickey Poisson bracket algebra of pseudo-differential operators, which has been studied earlier in the theory of integrable systems and is known to be isomorphic to the algebra of gauge-invariant differential polynomials [10] mentioned above.

It is worth noting that the differential operators which provide the bridge between the original Gelfand–Dickey construction and the KM approach to $\mathscr{W}$-algebras [10] (also constructed by an independent reasoning in [8]) appear naturally in our framework. They are nothing but the operators defining the differential equations satisfied by those (last row) components of g_L which are invariant under left gauge transformations. These differential equations can be obtained as a consequence of the obvious relation

$$(\kappa\,\partial_+ - J)\,g_L = 0, \tag{2.29}$$

where (2.29) is taken in the defining representation of the corresponding maximally non-compact real Lie algebra $\mathscr{G}$ (see Section V for more details).

In their review paper [10] Drinfeld and Sokolov studied the algebra of gauge-invariant differential polynomials by making use of the constrained KM algebra. We shall see that exploiting the full (unconstrained) embedding KM algebra yields further insight into the structure of classical $\mathscr{W}$-algebras and leads to new results.

III. THE $\mathscr{W}$-ALGEBRA

In this section we undertake a detailed analysis of the $\mathscr{W}$-algebra introduced in Section II. We first make the definition of the $\mathscr{W}$-algebra more explicit. The basic objects we are dealing with are gauge-invariant differential polynomials, W^i, defined on the space P of the constrained KM currents (i.e., currents J satisfying (2.8)). The Poisson brackets of the W^i are obtained by first extending their domain to the whole KM phase space, K, computing the Poisson brackets on K and then restricting to P. The Poisson brackets on K depend on the chosen extension of the W^i's (denoted by $\tilde{W}^i$), but their restrictions to P, which are again gauge invariant, do not. This follows by using the standard properties of the Poisson bracket from the first class nature of the constraints, and from the fact that the W^i's are invariant under the gauge transformations generated by the constraints (and from the assumption that the $\tilde{W}^i$ are real analytic in a neighbourhood of P).

There is no reason to expect that a generic extension of the W^i's closes under the Poisson bracket on K, but there is a trivial extension, which has the property that the Poisson brackets of the $\tilde{W}^i$'s not only close but have the same formal structure on K as on P, i.e.,

$$\{\tilde{W}^i(x), \tilde{W}^j(y)\} = \sum_k P_k^{ij}(\tilde{W})\, \delta^{(k)}(x^1 - y^1),$$

where the P_k^{ij}'s are the "structure differential polynomials" (2.28) of the $\mathscr{W}$-algebra. This particular extension is constructed as follows. First one expands the general KM current $J \in K$ in the Cartan–Weyl basis and notes that in P the upper triangular and Cartan components vary freely, while the lower triangular components are completely fixed by (2.8). The trivial extension $\tilde{W}^i$ of W^i is then defined to be the one which simply does not depend on the lower triangular current components.

Every element W of the $\mathscr{W}$-algebra generates canonical transformations on the KM phase space by the formula

$$J \to J + \delta_{\tilde{W}} J, \qquad \delta_{\tilde{W}} J = -\int_0^{2\pi} dx^1\, a(x) \{\tilde{W}(x), J\},$$

where $\tilde{W}(x)$ is any extension and $a(x)$ is an arbitrary test function. (Note that our equal-time Poisson brackets and spatial δ's are in fact equivalent to light-cone brackets and δ's. Prime everywhere means, even on δ's, "twice spatial-derivative" and this reduces to ∂_+ on quantities, $J(x)$, $W^i(x)$, and our test functions, which depend on $x=(x^0, x^1)$ through x^+ only.)

Since the transformation $\delta_{\tilde{W}}$ is canonical (preserves the KM-structure and hence the co-adjoint orbits in K), it follows that it can be represented as a field dependent KM transformation, i.e.,

$$\delta_{\tilde{W}} J = \delta_R J \equiv [R, J] + \kappa R',$$

where $R(J)$ is some (J-dependent) element of the KM algebra. The transformation $\delta_{\tilde{W}}$ transforms P into itself, and in fact induces a transformation δ^*_W on the space M of the gauge-orbits in P. The transformations $\delta_{\tilde{W}}$ corresponding to different extensions $\tilde{W}$ of W differ on P only by (field dependent) gauge transformations, and thus the induced transformation δ^*_W does not depend on the extension (only on W).

Of course, the reduced phase space M carries its own Poisson bracket structure which is inherited from the Poisson bracket structure of K, and is described by the standard Dirac bracket formula if one parametrizes M with some section of the gauge orbits in P (gauge choice). The induced $\mathcal{W}$-transformations δ^*_W are canonical transformations on M with respect to this induced (Dirac) Poisson bracket.

In Subsection III.1 we introduce some convenient gauges (called DS gauges), which will be used to show that the $\mathcal{W}$-algebra has a finite (l-dimensional) basis and to exhibit some particular bases W^i ($i=1 \cdots l$). The particular $\mathcal{W}$-generators W^i will be the gauge-invariant extensions (from the gauge section to P) of those current components (called DS currents) which survive the gauge fixing. Thus, in these gauges the $\mathcal{W}$-algebra appears as the Dirac bracket algebra of the DS currents. This is the basic fact on which most of our results are based.

In Subsection III.2 we exhibit a conformal field basis of the $\mathcal{W}$-algebra. In Subsection III.3, working in a DS gauge, we shall present an algorithm for finding the field dependent KM transformations which implement the induced $\mathcal{W}$-transformations δ^*_W. This algorithm is our main result since it enables us to calculate the action of the $\mathcal{W}$-algebra on any gauge-invariant quantity. In the last section we deal with some particular gauges which facilitate the study of some properties of the $\mathcal{W}$-algebra.

III.1. *Drinfeld–Sokolov Gauges*

In this section we recall the construction of a class of particularly convenient gauges in which the properties of the $\mathcal{W}$-algebra become apparent. This class of gauges has been introduced first by Drinfeld and Sokolov [10], so we call them DS gauges.

First we consider a special $sl(2, R)$ subalgebra of $\mathcal{G}$, $\mathcal{S}$, which will play an important role in what follows. This subalgebra is spanned by the Cartan element $\hat{\rho}$ in (2.25) and nilpotent generators $I_\pm$ such that

$$[I_+, I_-] = 2\hat{\rho}, \qquad [\hat{\rho}, I_\pm] = \pm I_\pm. \tag{3.1}$$

The step operators are explicitly given by

$$I_- = \sum_{i=1}^{l} \tau_i E_{-\alpha_i}, \qquad I_+ = \sum_{i=1}^{l} \frac{n_i}{\tau_i} E_{\alpha_i}, \tag{3.2a}$$

where

$$\tau_i = \frac{1}{2} \kappa \mu^i |\alpha_i|^2, \qquad n_i = 2 \sum_{j=1}^{l} (K^{-1})_{ij}. \tag{3.2b}$$

Note that since $\mathrm{Tr}(I_- E_{\alpha_i}) = \kappa \mu^i$ any element of P, i.e., any current fulfilling the constraints (2.8) (with $\mu^{\alpha_i} = \mu^i$), has the form

$$J(x) = I_- + \sum_{\alpha \in \Delta} \theta^\alpha(x) H_\alpha + \sum_{\varphi \in \Phi^+} \zeta^\varphi(x) E_\varphi. \tag{3.3}$$

The adjoint representation of $\mathscr{G}$ decomposes into $\mathscr{S}$ multiplets. Since $\hat{\rho}$ is an element of the Cartan subalgebra of $\mathscr{G}$ the step operators are $\hat{\rho}$-eigenstates,

$$[\hat{\rho}, E_\varphi] = h(\varphi) E_\varphi, \tag{3.4a}$$

where $h(\varphi)$ is the height of the root φ, i.e.,

$$h(\varphi) \equiv \sum_{i=1}^{l} m_i, \qquad \text{if} \quad \varphi = \sum_{i=1}^{l} m_i \alpha_i. \tag{3.4b}$$

Let $\mathscr{G}_h$ be the eigensubspace of $\hat{\rho}$ of eigenvalue h. If $h \neq 0$, then

$$\dim \mathscr{G}_h = \text{number of roots of height } h. \tag{3.5}$$

It can be shown [18, 10] that, if for $1 \leqslant h \leqslant h_\psi$ (h_ψ: height of the highest root ψ),

$$n_h = \dim \mathscr{G}_h - \dim \mathscr{G}_{h+1} \qquad \left(\sum_h n_h = l \right), \tag{3.6}$$

is not zero, then h is an exponent of $\mathscr{G}$ with multiplicity n_h.

We recall the meaning of the exponents and their multiplicities [18]: The ring of group-invariant polynomial functions on $\mathscr{G}$ is generated by l homogeneous elements whose degrees are determined by the exponents, h. More precisely, there are exactly n_h independent generators of order $h+1$. In other words, these generators define a complete set of independent Casimir operators. We note that $h=1$ and $h=h_\psi$ are always exponents. The multiplicity of the exponents is always 1, except for D_{2l}, where there are two independent Casimirs of order $2l$.

Note that for $(h \geqslant -1)$ I_- maps $\mathscr{G}_{h+1}$ into $\mathscr{G}_h$ injectively, that is

$$\dim I_-(\mathscr{G}_{h+1}) = \dim \mathscr{G}_{h+1}, \tag{3.7}$$

where $I_-(\mathscr{G}_{h+1}) = [I_-, \mathscr{G}_{h+1}]$. For any exponent, h, let V_h be a linear complement of $I_-(\mathscr{G}_{h+1})$ in $\mathscr{G}_h$ ($\dim V_h = n_h$) and let us also introduce the direct sum

$$V \equiv \bigoplus_h V_h \qquad (\dim V = l). \tag{3.8}$$

We choose a basis F_i ($i = 1, ..., l$) in V in such a way that

$$[\hat{\rho}, F_i] = h_i F_i \tag{3.9a}$$

holds, where

$$1 = h_1 \leqslant h_2 \leqslant \cdots \leqslant h_l = h_\psi \tag{3.9b}$$

is the list of the exponents with possible multiplicities included (see Appendix A).

The basic fact we need is that any constrained current of the form (3.3) can be uniquely gauge transformed into a current $\hat{J}(x)$ of the form

$$\hat{J}(x) = A(x)\, J(x)\, A^{-1}(x) + \kappa A'(x)\, A^{-1}(x) \equiv I_- + \sum_{i=1}^{l} W^i(x) F_i, \tag{3.10}$$

and that the $W^i(x)$ and the parameters $a^\varphi(x)$ of the gauge transformation

$$A(x) = \exp\left[\sum_{\varphi \in \Phi^+} a^\varphi(x) E_\varphi\right]$$

are differential polynomials in the components of $J(x)$. The proof of this statement [10] is actually easy. Using the fact that the gauge transformations are generated by upper triangular matrices, the inspection of (3.10) reveals that it is uniquely soluble in purely algebraic steps for both $W^i(x)$ and $a^\varphi(x)$ in terms of $J(x)$.

Denote now by M_V the space, whose "points" are currents of the form (3.10). The previous statement tells us that M_V defines a complete gauge fixing. Moreover, it also follows immediately that the components, $W^i(x)$, of the unique intersection point of M_V with the gauge orbit passing through $J \in P$ define gauge-invariant differential polynomials on P, which freely generate the $\mathscr{W}$-algebra. In other words, the W^i's form a basis in the algebra of gauge-invariant differential polynomials.

On the other hand, a completely general element of the KM algebra K can be expanded as

$$J(x) = \sum_{i=1}^{l} U^i(x) F_i + \sum_{\varphi \in \Phi^+} \xi^{-\varphi}(x) E_{-\varphi} + \sum_{\varphi \in \Phi^+} \xi^\varphi(x) [I_-, E_\varphi] \tag{3.11}$$

and M_V is obtained by first constraining the $\xi^{-\varphi}(x)$ by imposing (2.8) and then also fixing the residual gauge freedom by setting the $\xi^\varphi(x)$ to zero. The current components, $U^i(x)$, which are not affected by this two step restriction and the corresponding gauge-invariant differential polynomials, $W^i(x)$, are related by

$$U^i(x)_{|M_V} = W^i(x)_{|M_V}. \tag{3.12}$$

However, it should be stressed that conceptually the $U^i(x)$ (linear functions on K) and the $W^i(x)$ (gauge-invariant differential polynomials on P) are very different objects and must be carefully distinguished. To make this distinction even clearer we introduce a separate name for the U^i. From now on we shall refer to them as DS currents. It will turn out that most of our results are a consequence of (3.12). For example, this relation immediately implies that each differential polynomial $W^i(x)$ contains a leading term, i.e., a term without derivatives. In Subsection IV.1 we shall prove that the leading terms of any $\mathscr{W}$-basis are obtained by restricting Casimirs from K to P.

Now we discuss how the $\mathscr{W}$-algebra appears in a DS gauge. Clearly M_V inherits a Poisson bracket structure from the embedding KM algebra. This induced Poisson bracket structure is given by the familiar Dirac bracket formula [19]

$$\{f, g\}^* = \{f, g\} - \sum_{\alpha, \beta \in \Phi} \int_0^{2\pi} \int_0^{2\pi} dx^1\, dy^1 \{f, \xi^\alpha(x)\} \times D_{\alpha\beta}(x, y) \{\xi^\beta(y), g\}, \tag{3.13}$$

which is valid for two arbitrary phase space functions (f and g are functions on the KM phase space but only their restriction to M_V really matters). In this formula the ξ^α are the current components to be constrained (cf. (3.11)), and $D_{\alpha\beta}(x, y)$ is the inverse of

$$C^{\alpha\beta}(x, y) \equiv \{\xi^\alpha(x), \xi^\beta(y)\}, \tag{3.14}$$

which satisfies

$$\sum_{\beta \in \Phi} \int_0^{2\pi} dy^1 C^{\alpha\beta}(x, y)\, D_{\beta\gamma}(y, z) = \delta^\alpha_\gamma\, \delta(x^1 - z^1), \tag{3.15}$$

for arbitrary $\alpha, \gamma \in \Phi$. (Observe that the matrix-elements $C^{\alpha\beta}$, where $\alpha, \beta \in \Phi^-$, vanish on P, while the submatrix $C^{-\alpha\beta}$ ($\alpha, \beta \in \Phi^+$) is regular on M_V. Hence $C^{\alpha\beta}$ is also regular on M_V.)

Now the DS currents, $U^i(x)$, which survive the complete gauge fixing provide us with coordinates for the phase space M_V. Thus the induced Poisson structure of M_V can be described by specifying the Dirac brackets of the $U^i(x)$. The crucial point is that the Dirac brackets of the DS currents satisfy

$$\{U^i(x), U^j(y)\}^* = \{W^i(x), W^j(y)\} \quad \text{on} \quad M_V, \tag{3.16}$$

as a consequence of (3.12). As discussed earlier the Poisson brackets of the W^i's are in principle calculated by first extending them to K and then restricting the Poisson brackets calculated on K to P. Because of the gauge invariance of the W^i's, this is equivalent to calculating the Dirac brackets of the DS currents.

To summarize, we see that if the space of gauge orbits M is parametrized by the gauge section M_V, then its Poisson bracket structure is naturally described by means of the Dirac brackets of the DS currents, and that the $\mathscr{W}$-algebra can in fact be regarded as the Dirac bracket algebra of the DS currents. It will be demonstrated in the rest of this section that the properties of the $\mathscr{W}$-algebra are most effectively studied by making use of the DS gauges.

The family of DS gauges is parametrized by the possible choices of the linear space V in (3.8). It is easy to see that $U^1()x) \sim L(x)$ on M_V and therefore $W^1(x) \sim L(x)$ on the constraint-surface P, for any DS gauge.

III.2. *Conformal $\mathscr{W}$-Generators*

The energy–momentum density of the Toda theory, L in (2.25), generates the action of the conformal group on the KM phase space. This conformal action operates as

$$J \to J + \delta_L J$$
$$\delta_L J = -\int_0^{2\pi} dx^1\, a(x)\{L(x), J\} = (aJ)' + \kappa a''\hat{\rho} + a'[\hat{\rho}, J], \tag{3.17}$$

where $J \in K$ and $a(x)$ is any test function. The main point of this section is the observation that the $\mathscr{W}$-generators associated to a certain DS gauge (highest weight gauge) are primary fields with respect to this conformal action.

To demonstrate this it will be useful to describe the conformal action in terms of field dependent KM transformations. Let $R(J)$ be a KM algebra valued function defined on the KM phase space. Then it generates an infinitesimal (field dependent) KM transformation:

$$J \to J + \delta_R J, \qquad \delta_R J \equiv [R, J] + \kappa R'. \tag{3.18}$$

Now it is not difficult to verify that the conformal action δ_L is implemented by the field dependent KM transformation generated by the particular KM valued function

$$R_0(a, J) = \frac{1}{\kappa} aJ + a'\hat{\rho}, \tag{3.19a}$$

that is one has

$$\delta_L J = \delta_{R_0} J \qquad \text{for any} \quad J. \tag{3.19b}$$

The conformal action (3.17) transforms the set of constrained KM currents, P, into itself. Another crucial property of δ_L is that on P it commutes (modulo gauge transformations) with the action of the gauge transformations (2.22). Therefore (3.17) induces a conformal action on the gauge equivalence classes of the constrained currents, which amounts to an action on the set of gauge fixed currents;

M_V, representing those equivalence classes, for any choice of V. Our purpose below is to describe this induced conformal action

$$J \to J + \delta_L^* J \qquad (J \in M_V) \tag{3.20}$$

operating on M_V.

In general $J + \delta_L J \notin M_V$, and therefore to determine $\delta_L^* J$ we must find the compensating (unique) gauge transformation, $r = r(a, J)$, such that

$$J + \delta_L J + \delta_r J \in M_V, \qquad \text{for any} \quad J \in M_V, \tag{3.21a}$$

and then we have

$$\delta_L^* J = \delta_L J + \delta_r J = \delta_R J \qquad \text{with} \quad R = R(a, J) = R_0(a, J) + r(a, J). \tag{3.21b}$$

Before trying to determine $r(a, J)$ let us recall that δ_L^* is a canonical transformation on the reduced phase space M_V, generated by L by means of the Dirac bracket,

$$\delta_L^* J = -\int_0^{2\pi} dx^1\, a(x)\{L(x), J\}^* = -\int_0^{2\pi} dx^1\, a(x)\{U^1(x), J\}^* \tag{3.21c}$$

on M_V. Here the second equality holds provided we normalize the DS current U^1 according to

$$U^1(x) = L(x) \qquad \text{on} \quad M_V, \tag{3.22a}$$

which corresponds to the normalization of the basis vector F_1,

$$\operatorname{Tr} F_1 I_- = \kappa. \tag{3.22b}$$

With this normalization, as an obvious consequence of (3.16) and (3.21c), we have

$$\delta_L^* U^1 = a(U^1)' + 2a'U^1 - \kappa \operatorname{Tr}(\hat{\rho}^2) a'''. \tag{3.23}$$

Next we want to determine the induced conformal transformation of the U^i for $i \geqslant 2$. First, for an arbitrary gauge fixed current

$$J(x) = I_- + \sum_{i=1}^{l} U^i(x) F_i \tag{3.24}$$

one easily sees that

$$\delta_{R_0} J = \sum_{i=1}^{l} [a(U^i)' + (h_i + 1)\, a'U^i] F_i + \kappa a'' \hat{\rho}, \tag{3.25}$$

where h_i is the height of the Lie algebra element F_i according to (3.9a). The last term is "out of gauge" so one indeed needs a "compensating" gauge transformation.

In principle it is a purely algebraic problem to find $r(a, J)$, but in practice it is quite hard to produce an explicit formula for the solution in an arbitrary DS gauge for an arbitrary Lie algebra.

However, one can find a special gauge in which the form of $r(a, J)$ is particularly simple and the DS currents are primary with respect to the induced conformal action (3.21). The construction is based on the $sl(2, R)$ subalgebra $\mathcal{S}$ introduced in the previous section. Since the adjoint representation of $\mathcal{G}$ decomposes into $\mathcal{S}$ multiplets, it is natural to consider the corresponding highest weight states, i.e., those Lie algebra elements which commute with I_+. It is easy to see that the highest weight states in $\mathcal{G}_h$ span a natural complement of $I_-(\mathcal{G}_{h+1})$. Choosing this particular complement in the construction presented in Subsection III.1 we obtain a particular DS gauge, which we call the highest weight gauge. By using the fact that the basis vectors F_i of V in (3.8) now satisfy the condition

$$F_1 \sim I_+, \qquad [I_+, F_i] = 0, \qquad i = 2, ..., l, \tag{3.26a}$$

one easily proves that in the case of the highest weight gauge the compensating gauge transformation $r(a, J)$ is given by the simple formula

$$r(a, J) = -\tfrac{1}{2}\kappa a'' I_+. \tag{3.26b}$$

The corresponding conformal variation of the DS currents U^i then turns out to be

$$\delta_L^* U^i = a(U^i)' + (h_i + 1)\, a' U^i \qquad \text{for} \quad i = 2, ..., l, \tag{3.27}$$

i.e., they are indeed primary with respect to the induced conformal action (3.21). Equivalently, one can say that the corresponding gauge-invariant differential polynomials, W^i, are primary with respect to the original conformal action (3.17) (restricted to P). The conformal weights of the W^i's (U^i's) are $(h_i + 1)$, i.e., they are in one-to-one correspondence with the orders of the independent Casimirs of $\mathcal{G}$.

To summarize, we have proven that the generators W^i ($i = 2, ..., l$) defined by the highest weight gauge, together with $L = W^1$, constitute a natural, conformal field basis of the $\mathcal{W}$-algebra. This is one of our main results. As far as we know, an algorithm to find a conformal $\mathcal{W}$-basis has not been known before in the general case, although conformal $\mathcal{W}$-generators were explicitly exhibited for some particular low dimensional examples [6].

We now illustrate the idea of both the DS and the highest weight gauges on the example of $B_2 = o(3, 2)$. We use the convention [20] in which this Lie algebra consists of (5×5) matrices which are antisymmetric under reflection with respect to the "second diagonal." The Cartan subalgebra is spanned by the diagonal matrices in B_2. In this convention the Lie algebras of N and $\tilde{N}$ are represented by upper and lower triangular matrices, respectively. In particular, the E_α for $\alpha \in \Delta$ have non-zero entries only in the first slanted row above the diagonal. The Cartan element $\hat{\rho}$ in (2.25) is then easily found to be

$$\hat{\rho} = \text{diag}(2, 1, 0, -1, -2). \tag{3.28a}$$

By a convenient choice of the parameters τ_i in (3.2) we can choose the step operators of $\mathcal{S}$ as

$$I_+ = \begin{pmatrix} 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad I_- = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}. \tag{3.28b}$$

(Note that the value of the parameters τ_i is irrelevant since they can be redefined by rescaling the simple step-operators.) The elements of $\mathcal{G}_h$ are now those matrices in B_2 that have non-zero entries h steps above the diagonal only. Before describing the general DS gauge, we need to know the image $I_-(\mathcal{G}_2)$. In fact, an easy calculation yields that $I_-(\mathcal{G}_2)$ is the set of matrices of the form

$$\begin{pmatrix} 0 & x & 0 & 0 & 0 \\ 0 & 0 & -x & 0 & 0 \\ 0 & 0 & 0 & x & 0 \\ 0 & 0 & 0 & 0 & -x \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.29}$$

Since $\dim \mathcal{G}_1 = 2$, there is now a one parameter family of (one dimensional) linear sub-spaces V_1 of $\mathcal{G}_1$ which are complementary to $I_-(\mathcal{G}_2)$ in $\mathcal{G}_1$. These are nothing but the "lines" spanned by the vectors of the form

$$F_1 = F_1(p) = \begin{pmatrix} 0 & p & 0 & 0 & 0 \\ 0 & 0 & \kappa - p & 0 & 0 \\ 0 & 0 & 0 & p-\kappa & 0 \\ 0 & 0 & 0 & 0 & -p \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.30}$$

for any real p. Note that F_1 has been normalized according to (3.22b). (For the B_l algebras Tr means half of ordinary matrix trace in the defining representation.) The general current in the "DS gauge of parameter p" is written as

$$J(x) = I_- + U^1(x)F_1 + U^2(x)F_2 = \begin{pmatrix} 0 & pU^1 & 0 & U^2 & 0 \\ 1 & 0 & qU^1 & 0 & -U^2 \\ 0 & 1 & 0 & -qU^1 & 0 \\ 0 & 0 & -1 & 0 & -pU^1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}, \tag{3.31}$$

where $q \equiv \kappa - p$. We designate this set of gauge fixed currents as M_p. Observe that for $5p = 2\kappa$ the matrix F_1 is proportional to I_+, so that this value of p corresponds to the highest weight gauge.

It is not hard to calculate the compensating gauge transformation $r(a, J)$ in (3.21) which cancels the last term in (3.25). The reader can check that the result is

$$r(a, J) = -\tfrac{1}{2}\kappa a'' I_+ + \begin{pmatrix} 0 & 0 & y_2 & y_3 & 0 \\ 0 & 0 & 0 & 0 & -y_3 \\ 0 & 0 & 0 & 0 & -y_2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.32a}$$

with

$$y_2 = \lambda a''', \qquad y_3 = \lambda[\kappa a'''' - a'' U^1], \qquad \lambda = \kappa(2\kappa - 5p), \tag{3.32b}$$

which reduces to (3.26b) in the case of the highest weight gauge, as it should. The corresponding conformal variation of U^2 on M_p is given by

$$\delta_L^* U^2 = 4a' U^2 + a(U^2)' - \lambda((\kappa - p)\, a''' U^1 + \kappa(a'' U^1)' - \kappa^2 a'''''). \tag{3.33}$$

Since U^1 generates the induced conformal action on M_p through the Dirac bracket, from (3.33), taking (3.16) also into account, we can read off the Poisson bracket of W^1 with W^2 (restricted to P), which is now given as

$$\begin{aligned} \{W^1(x), W^2(y)\} &= 3W^2(x)'\, \delta(x^1 - y^1) + 4W^2(x)\, \delta'(x^1 - y^1) \\ &\quad + \lambda((p - \kappa)(W^1 \delta)''' - \kappa(W^1 \delta')'' + \kappa^2 \delta'''''). \end{aligned} \tag{3.34}$$

For $\lambda = 0$, that is for the highest weight gauge, the corresponding $\mathscr{W}$-generator, W^2, is a conformal primary field of weight 4. The generator $W^2 = W^2_{(p)}$ associated to any other DS gauge (of parameter p) transforms in a complicated, inhomogeneous manner under the conformal action.

III.3. *KM Implementation of $\mathscr{W}$-Transformations*

Here our purpose is to study the canonical transformations defined (as discussed at the beginning of the section) by the $\mathscr{W}$-algebra on the space of gauge orbits M. For this we consider the transformation δ_W^* induced on M by the following $\mathscr{W}$-transformation $\delta_{\tilde{W}}$ (acting originally on K),

$$J \to J + \delta_{\tilde{W}} J, \qquad \delta_{\tilde{W}} J = -\sum_{i=1}^{l} \int_0^{2\pi} dx^1\, a_i(x) \{\tilde{W}^i(x), J\}, \tag{3.35}$$

where the $\tilde{W}^i(x)$ are some arbitrary extensions from P to K of the $\mathscr{W}$-generators $W^i(x)$ associated to some DS gauge with gauge section M_V, and the $a_i(x)$ are arbitrary test functions. We parametrize M by M_V and in this parametrization the transformation δ^*_W is generated by means of the Dirac bracket according to

$$J \to J + \delta^*_W J, \qquad \delta^*_W J = -\{Q(a, J), J\}^*, \tag{3.36a}$$

with

$$Q(a, J) = \sum_{i=1}^{l} \int_0^{2\pi} dx^1\, a_i(x)\, U^i(x), \tag{3.36b}$$

where the $U^i(x)$ are the corresponding DS currents. Similar to the special case of the induced conformal transformation δ^*_L discussed in the preceeding section, the induced $\mathcal{W}$-transformation δ^*_W can be implemented by some field dependent KM transformation $R(a, J)$.

Of course, this KM implementation is in principle possible in any gauge, but here we show that in the DS gauges there exists a simple, effective algorithm for actually computing the KM valued function $R(a, J)$ which implements δ^*_W, i.e., which satisfies

$$\delta^*_W J = \delta_R J \qquad \text{for any} \quad J \in M_V. \tag{3.37}$$

This is immediately translated into the action of the $\mathcal{W}$-algebra on itself, since in the DS gauge the $W^i(x)$ reduce to the current components $U^i(x)$. An extra bonus of the KM implementation is that the KM algebra acts also on the G-valued WZNW field $g(x^+, x^-)$ and from that action we get

$$\delta^*_W g = \delta_R g, \qquad \text{that is,} \qquad \{Q(a, J), g\}^* = -R(a, J) \cdot g, \tag{3.38}$$

where "dot" means ordinary matrix product. From this equation we can read off the action of the $\mathcal{W}$-algebra on the Toda fields, which are the sub-determinants of g.

In order to make the presentation more concrete, we consider as examples the $\mathcal{W}$-algebras of the rank 2 Lie algebras A_2, B_2, and G_2. The A_2 example, which is the simplest non-trivial case, is included for the purpose of illustration. The B_2 example has some non-trivial features which will motivate some developments in subsequent subsections. Finally, G_2 (in Appendix B) illustrates the power of the method, since it enables us to compute the very non-trivial structure polynomials of this $\mathcal{W}$-algebra.

We start by presenting a general characterization of the *tangential* (gauge preserving) KM transformations for an arbitrary DS gauge. First we pick a point, $J_0 \in M_V$, and consider the tangential KM transformations at J_0. In other words, we want to describe all elements $R(J_0)$ of the KM algebra, which map $J_0 \in M_V$ into M_V, i.e., we want to solve the condition that

$$J_0 + \delta_R J_0 \equiv J_0 + [R, J_0] + \kappa R' \qquad \text{is in } M_V. \tag{3.39}$$

To give the general solution of this condition, it turns out to be useful to supplement the decompositions introduced in Subsection III.1,

$$\mathcal{G}_h = V_h \oplus I_-(\mathcal{G}_{h+1}), \qquad h \geqslant 1 \tag{3.40}$$

by similar ones for the subspaces $\mathcal{G}_{-h}$ of $\mathcal{G}$ corresponding to the negative roots (cf. (3.4)). Indeed, the decomposition we consider is induced by (3.40) as

$$\mathcal{G}_{-h} = V_{-h} \oplus U_{-h}, \qquad \text{for} \quad h \geqslant 1, \tag{3.41}$$

where V_{-h} is the transpose of V_h,

$$V_{-h} = \{v^t \mid v \in V_h\}, \tag{3.42a}$$

and U_{-h} is the annihilator of V_h in $\mathcal{G}_{-h}$ with respect to the scalar product Tr:

$$U_{-h} \equiv \{u \in \mathcal{G}_{-h} \mid \operatorname{Tr} uv = 0, \ \forall v \in V_h\}. \tag{3.42b}$$

(The transpose in (3.42a) can be defined abstractly by means of the Cartan–Weyl basis as $E^t_\varphi = E_{-\varphi}$, $H^t_\varphi = H_\varphi$, but in convenient conventions [20] it is the ordinary matrix transpose.)

Having introduced the necessary definitions we now return to the study of (3.39) and decompose the quantities entering this condition as

$$R(x) = \sum_{h \geqslant 1} (u_{-h}(x) + v_{-h}(x)) + \sum_{h \geqslant 0} y_h(x), \tag{3.43a}$$

where u_{-h}, v_{-h}, and y_h are in the subspaces U_{-h}, V_{-h}, and $\mathcal{G}_h$, respectively, and

$$J_0(x) = I_- + \sum_{h \geqslant 1} v^0_h(x), \qquad \delta_R J_0(x) = \sum_{h \geqslant 1} v_h(x), \tag{3.43b}$$

where both v^0_h and v_h must be in V_h. By analysing Eq. (3.39), one finds that if J_0 and all the $v_{-h}(x)$ are given, then the remaining components of R are uniquely determined differential polynomials in terms of these. Furthermore, it follows that the components of $\delta_R J_0$ are differential polynomials of J_0 and v_{-h} as well. In fact, the differential polynomials, R and $\delta_R J_0$, are linear in $v_{-h}(x)$, but in general nonlinear in J_0.

The above result provides us with a complete characterization of the tangential KM transformations at the arbitrarily chosen gauge-fixed current J_0. To actually prove this, one has to consider Eq. (3.39) height by height, starting from below, and use the following two properties of our Lie algebra decomposition:

First, for $h \geqslant 0$, I_- maps $\mathcal{G}_h$ into $\mathcal{G}_{h-1}$ in a one-to-one manner and this map is in fact onto for those h which are not exponents. Second, for $1 \leqslant h \leqslant (h_\psi - 1)$, I_- maps U_{-h} onto $\mathcal{G}_{-h-1}$ also in a one-to-one manner.

By using these properties of I_-, it is not difficult to verify that condition (3.39) is indeed uniquely soluble for $R(J_0)$ and $\delta_R J_0$ by purely algebraic means at every height, once J_0 and the v_{-h} are given, and that the solution is linear in v_{-h}.

Let us choose a basis $\{F^{-i}\}$ in $\bigoplus_h V_{-h}$ dual to the basis $\{F_i\}$ in $\bigoplus_h V_h$

$$F^{-i} \equiv \frac{F_i^t}{\operatorname{Tr} F_i F_i^t}. \tag{3.44a}$$

Since we assume that $F_i \in V_{h_i}$, the duality property (which we shall need later on)

$$\operatorname{Tr} F_i F^{-j} = \delta_i^j, \tag{3.44b}$$

is automatic in almost all cases, i.e., for those basis vectors which correspond to exponents $h_i \neq h_j$ of multiplicity 1, and we can also ensure this by a choice in the case of those two basis vectors which correspond to that exceptional exponent $h = (2l-1)$ of D_{2l} whose multiplicity is 2. Using this basis, we can now write R in (3.43) as

$$R = R(a, J_0) = \sum_{i=1}^{l} a_i(x) F^{-i} + \sum_{h \geq 1} u_{-h}(x) + \sum_{h \geq 0} y_h(x), \tag{3.45}$$

where the $a_i(x)$ are arbitrary functions and the u_{-h} and y_h are differential polynomials linear in the a_i, but not necessarily in J_0.

It is important to emphasize that, since J_0 was arbitrary in the construction, this equation defines an element $R(a, J)$ of the KM algebra for any J and a_i. According to its construction, at any fixed $J \in M_V$ this KM valued function $R(a, J)$ provides a parametrization of the set of tangential KM transformations at J, by the l arbitrary real functions $a_i(x)$. Hence it is clear that by varying J and at the same time promoting the parameters a_i to functionals of J one can write in the form $R(a(J), J)$ the most general field dependent, gauge-preserving KM transformation on M_V. So, in particular, the field dependent KM transformation implementing the induced $\mathscr{W}$-transformation δ_W^* (3.37) can also be written in this form with some functionals $a_i(J)$.

The result we prove is that the above constructed KM valued function $R(a, J)$ when considered for fixed (J-independent) a_i and varying J is the one which implements the induced $\mathscr{W}$-transformation δ_W^* according to (3.37) and (3.38).

This result means that we in effect replaced the task of finding the inverse of the matrix $C^{\alpha\beta}(x, y)$ (3.14) which enters the standard formula (3.13) of the Dirac bracket, by the much easier (as will be clear from the examples) task of solving Eq. (3.39).

To justify our claim we now show that

$$(\delta_R f)(J_0) = \{f, Q\}^* (J_0) \tag{3.46}$$

holds for an arbitrary real function $f(J)$, where $R = R(a, J)$ is given by the above construction, $Q = Q(a, J)$ is the moment of the DS currents defined in (3.36b) and

$J_0 \in M_V$ is arbitrary. To accomplish this we first recall that any element R_0 of the KM algebra defines a particular (field independent) KM transformation δ_{R_0} on the full KM phase space K, which is an (infinitesimal) canonical transformation generated by means of the KM Poisson bracket by the function

$$Q_0(J) = \int_0^{2\pi} dx^1 \operatorname{Tr} R_0(x)\, J(x). \tag{3.47}$$

This means that the relation

$$\delta_{R_0} F(J) = \{F, Q_0\}(J) \tag{3.48}$$

is satisfied on the full KM phase space K, for any real function $F(J)$. The trick is that now we take R_0 to be $R(a, J_0)$ in (3.45) for fixed J_0 and a. In this case we know that at J_0 the variation δ_{R_0} respects the constraints defining M_V ($R(a, J_0)$ was constructed by requiring this) and therefore at J_0 the constraint-contributions drop out from the Dirac-bracket of Q_0 (3.47) with any quantity. This way we derive

$$\{F, Q_0\}(J_0) = \{F, Q_0\}^* (J_0). \tag{3.49}$$

On the other hand, it is easy to see that for $R_0 = R(a, J_0)$ the functions $Q_0(J)$ (3.47) and $Q(a, J)$ (3.36b) differ on M_V only by a constant. This implies that they can be interchanged on M_V under Dirac-bracket. Taking this into account we immediately obtain (3.46) by combining (3.48) and (3.49) and by taking $F_{|M_V} = f_{|M_V}$. This finishes the proof.

We now illustrate on the simplest non-trivial example, $A_2 = sl(3, R)$, how to calculate the $\mathscr{W}$-algebra by our algorithm. The $\mathscr{W}(A_2)$-algebra is well known but it is worth reconsidering it in the present framework as an illustration. We use again the conventions of [20]. The Cartan element of the speical $sl(2, R)$ is represented by

$$\hat{\rho} = \operatorname{diag}(1, 0, -1). \tag{3.50a}$$

Choosing $\tau_1 = \tau_2 = 1$ in (3.2) the remaining generators of $\mathscr{S}$ are given by

$$I_+ = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad I_- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \tag{3.50b}$$

As in the B_2 example, there is a one parameter family of DS gauges, and the gauge fixed current in the "DS gauge of parameter p" is written as

$$J(x) = I_- + U^1(x) F_1 + U^2(x) F_2, \tag{3.51}$$

where we can take

$$F_1 = F_1(p) = \begin{pmatrix} 0 & p & 0 \\ 0 & 0 & \kappa - p \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad F_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.52}$$

Here F_1 is normalized according to (3.22b). The highest weight gauge corresponds to $2p = \kappa$, but here we choose to work in the "Wronskian gauge" $p = \kappa$, which is the gauge usually considered in the literature [5–9] (the origin of the name "Wronskian gauge" will become clear in Section V).

Our aim is to find the explicit form of $R(a, J)$ in (3.45) in the Wronskian gauge. In this case

$$F^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ 1/\kappa & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad F^{-2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tag{3.53}$$

and U_{-1} in (3.42b) now consists of matrices for which only a_{32} is non-zero, while U_{-2} is trivial. The explicit form of R in (3.45) reads then as

$$R = R(a_1, a_2, J) = \begin{pmatrix} y_0 & y_1 & y_2 \\ a_1/\kappa & (\tilde{y}_0 - y_0) & \tilde{y}_1 \\ a_2 & u_{-1} & -\tilde{y}_0 \end{pmatrix}, \tag{3.54}$$

where the a_i are arbitrary functions and the other entries are to be determined by the condition that the variation $\delta_R J$ must leave J "form invariant." In our case this means that $\delta_R J$ must be of the form

$$\delta_R J = [R, J] + \kappa R' \equiv \begin{pmatrix} 0 & \kappa\,\delta U^1 & \delta U^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.55}$$

since in the Wronskian gauge

$$J = \begin{pmatrix} 0 & \kappa U^1 & U^2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \tag{3.56}$$

As is follows from our general result, substituting (3.54) and (3.56) into (3.55) one obtains a system of equations which is uniquely soluble in purely algebraic steps for both the component functions $u_{-1} \cdots y_2$ of R and for the corresponding variation of J. One has to consider (3.55) height by height, starting from below, and easily obtains the following formulae for the components of $R(a, J)$:

$$u_{-1}=\frac{a_1}{\kappa}-\kappa a_2', \qquad y_1=a_1 U^1+a_2 U^2-\kappa y_0',$$
$$y_0=a_1'+\frac{\kappa}{3}[a_2 U^1-\kappa a_2''], \qquad \tilde{y}_1=a_2 U^2-\kappa\tilde{y}_0', \tag{3.57}$$
$$\tilde{y}_0=2y_0-a_1', \qquad y_2=\frac{a_1}{\kappa}U^2+\kappa\tilde{y}_1'.$$

Before proceeding let us note that R $(a_2=0)$ implements the induced conformal action in the Wronskian gauge, and in fact one can rewrite the above formula as

$$R(a_1, a_2=0, J)=\left[\frac{1}{\kappa}a_1 J+a_1'\hat{\rho}\right]-\kappa\left[\frac{1}{2}a_1'' I_+ +\kappa a_1''' F_2\right], \tag{3.58}$$

which is consistent with (3.21) and (3.19a) describing the conformal action in general. The variation of J under the KM transformation δ_R is found to be

$$\begin{aligned}\delta U^1 &= [a_1(U^1)'+2a_1' U^1-2\kappa a_1'''] \\ &\quad+[2a_2(U^2)'+3a_2' U^2-\kappa^2(a_2 U^1)''+\kappa^3 a_2'''']\end{aligned} \tag{3.59}$$

and

$$\begin{aligned}\delta U^2 &= [a_1(U^2)'+3a_1' U^2+\kappa^2 a_1'' U^1-\kappa^3 a_1''''] \\ &\quad+a_2[\kappa^2(U^2)''+\tfrac{2}{3}\kappa^3 U^1(U^1)'-\tfrac{2}{3}\kappa^4(U^1)'''] \\ &\quad+a_2'[\tfrac{2}{3}\kappa^3(U^1)^2+2\kappa^2(U^2)'-2\kappa^4(U^1)''] \\ &\quad-2\kappa^4 a_2''(U^1)'-\tfrac{4}{3}\kappa^4 a_2''' U^1+\tfrac{2}{3}\kappa^5 a_2'''''.\end{aligned} \tag{3.60}$$

Now by combining Eqs. (3.36) and (3.37), it follows that

$$\delta U^i(x)=\sum_{j=1,2}\int_0^{2\pi} dy^1 a_j(y)\{U^i(x), U^j(y)\}^* \tag{3.61}$$

holds, so from (3.59) and (3.60) one can read off the Dirac brackets of the DS currents, yielding immediately the Poisson brackets of W^1 and W^2 according to (3.16). (See Subsection IV.1.)

Observe that the W^2 generator associated to the Wronskian gauge is not a primary field with respect to $W^1=L$. However, it is easy to see that the combination

$$W^2-\frac{\kappa^2}{2}(W^1)' \tag{3.62}$$

defines a primary field of weight 3. By investigating the transformation rules between the $\mathscr{W}$-bases corresponding to different DS gauges one can prove that (3.62) is precisely the $\mathscr{W}$-generator associated to the highest weight gauge.

Note also that in this example the components of $R(a, J)$ in (3.57) are only linear functions of the current components, and as a consequence $\delta_R J$ is at most quadratic in J, which implies that the Poisson brackets of the $\mathscr{W}$-generators are also (at most) quadratic polynomials. This is not always the case, as can be seen, e.g., in the example of B_2.

We now illustrate the action of the $\mathscr{W}$-generators on the components of the matrix-valued field $g(x^+, x^-)$ on this example. All we have to do is to use the results (3.57) for (3.54) and substitute this $R(a, J)$ into (3.38).

Let us discuss the conformal transformations first. For this case, we find

$$\delta_1 g_{1i} = a_1 g_{1i} + (a_1 U^1 - \kappa a_1'') g_{2i} + \left(\frac{a_1}{\kappa} U^2 - \kappa^2 a_1'''\right) g_{3i} \tag{3.63a}$$

$$\delta_1 g_{2i} = \frac{a_1}{\kappa} g_{1i} - \kappa a_1'' g_{3i} \tag{3.63b}$$

$$\delta_1 g_{3i} = \frac{a_1}{\kappa} g_{2i} - a_1' g_{3i}. \tag{3.63c}$$

To simplify (3.63) we can make use of the relation between the currents and the matrix-valued fields, (2.6). In this example this gives

$$g_{2i} = \kappa\, \partial\psi_i, \qquad g_{1i} = \kappa^2\, \partial^2\psi_i, \tag{3.64}$$

where $\psi_i = g_{3i}$ and $\partial = \partial/\partial x^+$. Equation (2.6) also gives a differential equation satisfied by ψ_i (see Section V), which we will not explicitly use here. Using (3.64), (3.63c) simplifies to

$$\delta_1 \psi_i = a_1\, \partial\psi_i - a_1' \psi_i, \tag{3.65}$$

which tells us that ψ_i is a primary field with conformal spin -1, whereas the remaining equations in (3.63) describe the conformal transformation properties of the secondary fields (3.64).

We now turn to the genuine $\mathscr{W}$-transformation generated by U^2. Using (3.64) again, we find

$$\delta_2 \psi_i = a_2 \left(\kappa^2\, \partial^2 - \frac{2\kappa}{3} U^1\right) \psi_i - \kappa^2 a_2'\, \partial\psi_i + \frac{2\kappa^2}{3} a_2'' \psi_i. \tag{3.66}$$

Equation (3.66) can be thought of as the transformation rule for a "$\mathscr{W}$-primary" field under the W^2-transformation (for the A_2 $\mathscr{W}$-algebra).

For the algebra B_2 we have derived the conformal action in Subsection III.2. Thus it only remains to determine the canonical transformation generated by W^2

to know the complete set of transformations generated by the $\mathcal{W}$-algebra in this case, from which we can of course again (as for A_2) read off the $\mathcal{W}$-relations themselves. By applying the algorithm presented above one finds after lengthy but straightforward calculations that

$$\{U^2(x), U^2(y)\}^* = \frac{1}{2} \sum_{i=0}^{2} [\mathcal{F}^{2i+1}(x) + \mathcal{F}^{2i+1}(y)] \times \delta^{(2i+1)}(x^1 - y^1) - \kappa^5 P\, \delta^{(7)}(x^1 - y^1) \tag{3.67}$$

on M_p, where

$$\begin{aligned} \mathcal{F}^1 &= Q_1^1(U^2)'' + Q_2^1 U^1 U^2 + Q_3^1(U^1)'''' + Q_4^1 U^1 (U^1)'', \\ &\quad + Q_5^1((U^1)')^2 + Q_6^1(U^1)^3, \\ \mathcal{F}^3 &= Q_1^3 U^2 + Q_2^3(U^1)'' + Q_3^3(U^1)^2, \qquad \mathcal{F}^5 = Q^5 U^1. \\ P &= p^2 + (p-q)^2, \qquad \text{and} \qquad q = \kappa - p. \end{aligned} \tag{3.68}$$

Here Q^5, Q_k^j are polynomials of the parameter p, given explicitly as

$$\begin{aligned} &Q_1^1 = -2\kappa^2 p, \qquad Q_2^1 = 8p^2 - 16\kappa p + 4\kappa^2, \qquad Q_3^1 = 2\kappa^4[P + 2pq] \\ &Q_1^3 = 2\kappa^2(3p - q), \qquad Q_2^3 = -2\kappa^4[2P + 3pq], \qquad Q_6^1 = 2(q+\kappa)\, pq^2 \end{aligned} \tag{3.69a}$$

$$\begin{aligned} Q_3^3 &= -\kappa(q+\kappa)^2 P - 2\kappa^2(2q+\kappa)\, pq \\ Q^5 &= 2\kappa^3[(q+\kappa)P + \kappa pq] \\ Q_4^1 &= 2\kappa^2(q+\kappa)P + 2\kappa^2(q+2\kappa)\, pq \\ Q_5^1 &= \kappa[3q^2 + 4\kappa q + 2\kappa^2]P + 2\kappa^2(q+2\kappa)\, pq. \end{aligned} \tag{3.69b}$$

Observe that unlike for the A_2-model, there is now also a cubic term, $(U^1)^3$ in $\mathcal{F}^1$. The coefficient Q_6^1 of this single cubic term vanishes in the special cases when $p = 0$, κ or $p = 2\kappa$. In other words, the B_2 $\mathcal{W}$-algebra is given by quadratic relations in that $\mathcal{W}$-bases which are associated to the particular DS gauges of parameter $p = 0$, κ or $p = 2\kappa$. These "quadratic gauges" could be useful in the quantization of the $\mathcal{W}$-algebra, since normal ordering is more complicated when the order of the polynomials involved gets larger. In contrast, the conformal properties are hidden in these gauges and are not as transparent as in the highest weight gauge (which belongs to $5p = 2\kappa$ in the B_2 example).

III.4. *Other Convenient Gauges*

In the previous subsections we have discussed the DS type gauges and have shown that choosing a DS gauge naturally leads to a corresponding choice of basis for the $\mathcal{W}$-algebra, by relating the $\mathcal{W}$-generators to the non-vanishing current components in that gauge. The highest weight gauge plays a particular role because the corresponding $\mathcal{W}$-generators are conformal primary fields (with the exception of the conformal generator W^1). In the examples of A_2 and B_2 (C_2) we have shown that it is possible to choose such DS gauges in which the generating relations of the $\mathcal{W}$-algebra are quadratic. These gauges are also important because the quadratic closure of the algebra simplifies the quantization. In this section we show that such gauges exist for the algebras A_l, B_l, and C_l. We will see in the next section that they are not available for the rest of the Lie algebras.

We start by considering A_l, i.e., $sl(l+1, R)$ and will use the defining representation. Here (and also for B_l, C_l, and D_l later) we shall use the conventions [20] in which the positive and negative step-operators are upper and lower triangular matrices, respectively, and the elements of the Cartan subalgebra are diagonal matrices. For simplicity, now we choose all τ_i in (3.2) to be equal to 1, and then the matrix I_- reads

$$I_- = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & & & \ddots & \vdots & \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}. \tag{3.70}$$

The elements of $\mathcal{G}_h$ are matrices with non-zero entries only in the slanted row h steps above the diagonal. The image $I_-(\mathcal{G}_{h+1})$ (for $h \geqslant 0$) consists of those matrices in $\mathcal{G}_h$ for which the sum of the matrix elements is zero. Fixing a DS gauge means choosing a single matrix in $\mathcal{G}_h$ for which the sum of the matrix elements is different from zero. The simplest choice yields the "*Wronskian*" *gauge* defined by

$$J = I_- + \begin{pmatrix} 0 & U^1 & \cdots & U^l \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{3.71}$$

This gauge is a special example of the more general *block gauges* for which

$$J = I_- + j = I_- + \begin{pmatrix} 0 & U \\ 0 & 0 \end{pmatrix} \tag{3.72}$$

and U is a $p \times q$ block ($p+q = l+1$) containing the l DS currents. The "Wronskian" gauge is the special case when $p=1$ and $q=l$. In general these "block" gauges are not unique: we are still free to distribute the DS currents in a number of different ways along the intersections of the slanted rows with the block.

Now we are going to show that the $\mathcal{W}$-algebra closes quadratically in any of these "block" gauges. According to the results developed in the previous subsection, we can derive the $\mathcal{W}$-relations by determining the field dependent KM transformation $R(a, J)$ in (3.45) which implements the induced $\mathcal{W}$-transformations on M_V. To do this we first rewrite the defining Eq. (3.39) of $R(a, J)$ in the form

$$[R, I_-] + \kappa R' = \delta J + [j, R]. \tag{3.73}$$

Now, since we know that the unique solution of (3.73) for $R = R(a, J)$ and δJ is linear in the infinitesimal parameters of the transformation, i.e., in the functions a_i introduced in (3.45), and polynomial in the given gauge fixed current $I_- + j$, we can expand both R and δJ in powers of the DS currents (j),

$$\begin{aligned} R &= R_0 + R_1 + R_2 + \cdots \\ \delta J &= (\delta J)_0 + (\delta J)_1 + (\delta J)_2 + \cdots \end{aligned} \tag{3.74}$$

and solve (3.73) perturbatively,

$$[R_m, I_-] + \kappa R'_m = (\delta J)_m + [j, R_{m-1}], \qquad m = 0, 1, 2, ... \tag{3.75}$$

Since in the "block" gauge both J and δJ are upper triangular in the block sense,

$$j = \begin{pmatrix} 0 & U \\ 0 & 0 \end{pmatrix}, \qquad \delta j = \begin{pmatrix} 0 & \delta U \\ 0 & 0 \end{pmatrix}. \tag{3.76}$$

If we write out (3.73) in "block" components it is not difficult to see that the first order solution must be of the form

$$R_1 = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}, \tag{3.77}$$

where the $p \times p$ block A and the $q \times q$ block C are further restricted by

$$A_{pi} = 0 \quad \text{for} \quad i \leqslant p-1 \quad \text{and} \quad C_{i1} = 0 \quad \text{for} \quad i \geqslant 2 \tag{3.78}$$

and that the second order solution is of the form

$$R_2 = \begin{pmatrix} 0 & D \\ 0 & 0 \end{pmatrix} \quad \text{with} \quad D_{i1} = D_{pj} = 0, i = 1, 2, ..., p; j = 1, 2, ..., q. \tag{3.79}$$

For the "block" gauges the expansion stops here and, by the results of Subsection III.3, this implies that the algebra of the $\mathcal{W}$-generators corresponding to any DS gauge from the family of block gauges closes quadratically indeed.

Note that the "Wronskian" gauge is special since $D = 0$ in this case and thus the KM transformation $R = R(a, j)$ is only linear in the DS currents. The algebra is still quadratic, since

$$(\delta J)_2 = [R_1, j]. \tag{3.80}$$

For the other matrix algebras, B_l and C_l, one can define analogous "block" gauges by embedding them into appropriate A-type algebras.

For $C_l \sim sp(2l, R)$ we can use the $2l$-dimensional defining representation. We write the C_l matrices in terms of four $l \times l$ square blocks. In this notation the symplectic metric is given by

$$G=\begin{pmatrix} 0 & \varepsilon \\ -\varepsilon & 0 \end{pmatrix}, \tag{3.81}$$

where the only nonvanishing entries of ε are in the second diagonal (the diagonal from bottom-left to top-right), and these entries are all 1. The elements of the Lie algebra are represented by matrices of the form

$$K=\begin{pmatrix} A & B \\ C & -\tilde{A} \end{pmatrix}, \qquad \text{where} \quad \tilde{B}=B, \tilde{C}=C \tag{3.82}$$

and $\tilde{}$ means reflection with respect to the second diagonal.

Positive (negative) step-operators are again upper (lower) triangular matrices and elements of the Cartan subalgebra are diagonal. By a convenient choice of the (irrelevant) parameters τ_i, I_- is now given by the $2l \times 2l$ matrix:

$$I_-=\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 0 \end{pmatrix}. \tag{3.83}$$

It has l 1 entries and $(l-1)$ (-1)'s.

The "block" gauges, in which the algebra closes quadratically are characterized by

$$J=I_- + \begin{pmatrix} 0 & U \\ 0 & 0 \end{pmatrix}, \tag{3.84}$$

where $\tilde{U}=U$ and it has non-vanishing components along every second slanted row, corresponding to the exponents of this algebra.

Finally, for $B_l \sim so(l+1, l)$ we take the $(2l+1)$-dimensional vector representation. In a 3×3 block matrix notation corresponding to the partition $l+1+l$ the Lorentzian metric is

$$G=\begin{pmatrix} 0 & 0 & \varepsilon \\ 0 & 1 & 0 \\ \varepsilon & 0 & 0 \end{pmatrix} \tag{3.85}$$

and the elements of the Lie-algebra are of the form

$$K=\begin{pmatrix} A & X & B \\ Y^t & 0 & -X^t \\ C & -Y & -\tilde{A} \end{pmatrix}, \qquad \text{where} \quad \tilde{B}=-B, \tilde{C}=-C. \tag{3.86}$$

The matrix I_- is again similar to (3.83) but it is now a $(2l+1)\times(2l+1)$ matrix and has l upper entries 1 and l lower entries (-1). The "block" gauges for this algebra are defined by

$$J=I_-+\begin{pmatrix} 0 & x & b \\ 0 & 0 & -x^t \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.87}$$

where $\tilde{b}=-b$ and the DS currents are again distributed along every second slanted row.

An other convenient gauge is what we will call the diagonal gauge. It is defined by

$$J(x)=I_-+\sum_{i=1}^{l}\theta_i(x)H_i. \tag{3.88}$$

(Here we choose the $\{H_i\}$ to form an orthonormal basis for the Cartan subalgebra.) Note that this is a new type of gauge fixing, not a member of the family of the DS gauges, but it will turn out to be very useful in applications and it is most useful in the quantum theory. Before we start discussing the gauge choice (3.88) in detail, we mention two difficulties connected with it. We will illustrate these difficulties on the simplest example, $sl(2, R)$.

In this case the gauged fixed current in the diagonal gauge is parametrized by a single real field $\theta(x)$,

$$J_{\text{diag}}=\begin{pmatrix} \theta & 0 \\ 1 & -\theta \end{pmatrix} \tag{3.89}$$

and it is easy to see that the transformation from the "Wronskian" gauge

$$J_{\text{Wron}}=\begin{pmatrix} 0 & U(x) \\ 1 & 0 \end{pmatrix} \tag{3.90}$$

to the diagonal gauge amounts to solving the Riccati equation

$$\theta^2-\kappa\theta'=U. \tag{3.91}$$

Now, if we require all fields to be periodic and integrate the Riccati equation over the period, the derivative term drops out and we see that (3.91) has no solution unless

$$\int_0^{2\pi} U(x)\, dx^1 \geqslant 0. \tag{3.92}$$

In other words, the diagonal gauge can only be reached from that part of the phase space where (3.92) is satisfied.

A related difficulty is that when the Riccati equation can be solved, its solution is not unique, it in fact has two independent solutions. (For an arbitrary Lie algebra, the number of independent solutions of the analogous equations is equal to the order of the Weyl-group.)

However, note that when available the diagonal gauge is locally well defined (the ambiguities mentioned above correspond to finite gauge transformations) and therefore the corresponding Dirac brackets are also well defined. Since the $\mathscr{W}$-algebra is determined by polynomial relations, its structure can be analysed by restricting the considerations to that part of the phase space where the diagonal gauge is available and we will see that this is often convenient.

Expanding the general KM current, J, in the Cartan–Weyl basis as

$$J = \sum_{\varphi \in \Phi^+} \xi^{-\varphi} E_{-\varphi} + \sum_{i=1}^{l} \theta_i H_i + \sum_{\varphi \in \Phi^+} \zeta^{\varphi} E_{\varphi}, \tag{3.93}$$

the set of constraints defining the diagonal gauge can naturally be divided into two parts,

$$\chi = \begin{pmatrix} \xi \\ \zeta \end{pmatrix}. \tag{3.94}$$

The diagonal gauge is defined by constraining the $\xi^{-\varphi}$ by imposing the original constraints (2.8) and, in addition, setting the ζ^{φ} to 0. Since on the corresponding constraint surface

$$\{\xi, \xi\} = 0 \qquad \text{and} \qquad \{\zeta, \zeta\} = 0, \tag{3.95}$$

the C operator, whose inverse enters the formula for the Dirac bracket can schematically be written as

$$C = \{\chi, \chi\} \approx \begin{pmatrix} 0 & B \\ -B & 0 \end{pmatrix}, \tag{3.96}$$

where $B = \{\xi, \zeta\}$. Now the Dirac bracket of any two quantities u and v takes the form

$$\{u, v\}^* = \{u, v\} + \{u, \xi\}\, B^{-1}\{\zeta, v\} - \{u, \zeta\}\, B^{-1}\{\xi, v\}. \tag{3.97}$$

The important property of the diagonal fields $\theta_i(x)$ that makes the diagonal gauge extremely simple is that they (weakly) commute with the additional constraints ζ^{φ}:

$$\{\theta_i(x), \zeta\} \approx 0. \tag{3.98}$$

Because of (3.98), the Dirac bracket of two diagonal currents is the same as their original KM Poisson bracket:

$$\{\theta_i(x), \theta_j(y)\}^* \approx \{\theta_i(x), \theta_j(y)\} = \kappa\, \delta_{ij}\, \delta'(x^1 - y^1). \tag{3.99}$$

In other words, the diagonal components of the current are a set of free fields. Therefore in the diagonal gauge the $\mathscr{W}$-generators are given as differential polynomials in free fields and these differential polynomials are simply obtained by restricting the full (gauge-invariant) differential polynomials to the "diagonal currents" of the form (3.88). This free-field representation of the $\mathscr{W}$-generators is called the Miura-transformation and has been used to quantize the theory [5].

IV. Casimir Algebra

IV.1. *Leading Terms and Casimir Algebra*

We have already seen that any DS gauge defines a basis of the $\mathscr{W}$-algebra, and that there is a one-to-one correspondence between the conformal weights of the $\mathscr{W}$-generators associated to the highest weight gauge (or the scale dimensions of the $\mathscr{W}$-generators associated to any DS gauge) and the orders of the independent Casimirs of the underlying simple Lie algebra. In this section we shall elaborate on this connection further, by showing that the leading terms of the $\mathscr{W}$-generators (i.e., terms without any derivatives) are always Casimirs (restricted to P). Then we demonstrate that the Casimirs themselves form a polynomial algebra under the Poisson bracket, which is a truncated version of the full $\mathscr{W}$-algebra. This Casimir algebra, in its quantum version, has been studied in [15].

We shall denote the leading terms of the $\mathscr{W}$-generators, W^j, by W^j_0. Since these leading terms contain no derivatives, they are invariant under rigid gauge transformations, that is

$$W^j_0(J^A) = W^j_0(J) \qquad \text{for} \quad A \in N, \text{ where } J^A = AJA^{-1}$$

for any constrained current ($J \in P$). On the other hand, an arbitrary Casimir C^j is a group-invariant polynomial, that is, for any KM current J and an arbitrary $B \in G$ one has

$$C^j(J^B) = C^j(J), \qquad \text{where} \quad J^B = BJB^{-1}.$$

First we want to show that the leading terms of the $\mathscr{W}$-generators are restricted Casimirs, or in other words that

$$W_0^j(J) = C^j(J), \qquad J \in P \tag{4.1}$$

for some C^j.

To do this we shall use the theorem of Chevalley from the theory of invariant polynomials [18], which we now recall. This theorem states that there is a one-to-one correspondence between the Casimirs and the Weyl-invariant polynomials on the Cartan subalgebra, and that the correspondence is simply given by restriction. That is, first, if $C^j(J)$ is an arbitrary group invariant polynomial (Casimir) on $\mathscr{G}$, then its restriction to the Cartan subalgebra, $\bar{C}^j(H)$, is a Weyl-invariant polynomial. (We shall denote the Cartan subalgebra by $\mathscr{H}$ and the restriction of any function to $\mathscr{H}$ by an overbar.) Conversely, from any given Weyl-invariant polynomial on $\mathscr{H}$, a corresponding full group invariant can be reconstructed in a unique way.

For later use we also recall that the uniqueness of the reconstruction is proven by "diagonalization." First note that for any Lie algebra element J in the compact form of $\mathscr{G}$ there exists a group element $g \in G$ that "diagonalizes" J:

$$J^g = gJg^{-1} = H(J) \in \mathscr{H}.$$

(The use of the compact form is justified here since the problem is purely algebraic.) Using the group invariance of the Casimir C^j we see that

$$C^j(J) = C^j(J^g) = C^j(H(J)) = \bar{C}^j(H(J))$$

so $\bar{C}^j$ determines the full Casimir C^j uniquely indeed.

By using Chevalley's theorem, (4.1) will follow if we can prove that the restriction of $W_0^j(J)$ to currents J in the diagonal gauge (cf. (3.88)) is a Weyl-invariant polynomial of the Cartan components of J. To do this we only have to show that for any "diagonal" constrained current J it its possible to find such rigid gauge transformations $A \in N$, whose action on the Cartan components θ_i of J coincide with the action of the Weyl-group on the θ_i.

To show this, let us choose a simple root α_k and consider the action of the finite gauge transformation

$$A = e^a \qquad \text{with} \quad a = \omega E_{\alpha_k} \tag{4.2a}$$

on a constrained current $J \in P$,

$$J \to J^{(a)} = e^a J e^{-a} = J + [a, J] + \tfrac{1}{2}[a, [a, J]] + \cdots, \tag{4.2b}$$

where ω is an arbitrary real parameter. Parametrizing the constrained current $J \in P$

$$J = I_- + \sum_{i=1}^{l} \theta_i H_{\alpha_i} + \sum_{i=1}^{l} \zeta_i E_{\alpha_i} + \sum_{\varphi} \zeta_\varphi E_\varphi,$$

where φ runs over the set of positive non-simple roots, we find (remember that I_- is given by (3.2)) that the components of J transform under (4.2) as

$$\theta_i^{(a)} = \theta_i + \omega\,\delta_{ik}\tau_k$$

$$\zeta_i^{(a)} = \zeta_i - \frac{2\omega}{|\alpha_k|^2}\,\delta_{ik}\sum_j (\alpha_j, \alpha_k)\theta_j - \omega^2\,\delta_{ik}\tau_k$$

$$\zeta_\varphi^{(a)} = \zeta_\varphi + \sum_j \Psi_{\varphi j}(\omega)\zeta_j + \sum_{\varphi'} \Phi_{\varphi\varphi'}(\omega)\zeta_{\varphi'},$$

where the precise form of the coefficients $\Psi_{\varphi j}$ and $\Phi_{\varphi\varphi'}$ is irrelevant for our purpose. Now we fix $J \in P$ and choose the parameter ω to be

$$\omega = -\frac{2}{\tau_k\,|\alpha_k|^2}\sum_j (\alpha_j, \alpha_k)\theta_j,$$

so that the set of components (ζ_i, ζ_φ) transforms homogeneously as

$$\zeta_i^{(a)} = \zeta_i, \qquad \zeta_\varphi^{(a)} = \zeta_\varphi + \sum_j \Psi_{\varphi j}\zeta_j + \sum_{\varphi'} \Phi_{\varphi\varphi'}\zeta_{\varphi'},$$

which implies that the transformation (4.2) applied to the "diagonal" current

$$J_{\text{diag}} = I_- + \sum_{i=1}^{l} \theta_i H_{\alpha_i}$$

takes it into another current which is also in the diagonal gauge. Moreover, with this choice of ω the action of the gauge transformation $A = e^a$ (4.2a) on the the Cartan components θ_i of this particular diagonal current is

$$\theta_i^{(a)} = \theta_i - \frac{2}{|\alpha_k|^2}\,\delta_{ik}\sum_j (\alpha_j, \alpha_k)\theta_j,$$

which is precisely the same as the effect of the Weyl-reflection corresponding to the simple root α_k on the Cartan components θ_i. This implies that every Weyl-transformation of the Cartan components of the diagonal currents can indeed be implemented by rigid gauge transformations. (Since the Weyl-group is not a subgroup of N, the particular rigid gauge transformation A which "implements" a given Weyl-transformation on the components θ_i of a "diagonal" current J_{diag} must depend on the particular current on which it acts, and is really field-dependent according to the above construction.) Since the leading term W_0^j is invariant under rigid gauge transformations, it follows that its restriction $\bar{W}_0^j$ to the diagonal gauge is a Weyl-invariant polynomial of the current components θ_i. Chevalley's theorem then tells us that $\bar{W}_0^j$ is the restriction of a uniquely determined Casimir C^j to the diagonal currents (note that I_- has no contribution in $C^j(J_{\text{diag}})$ because of the neutrality of the group invariant C^j). To finish the proof of (4.1) one has to show that the leading term W_0^j itself is the restriction of the same Casimir C^j to P. This last step follows from the fact that W_0^j and the restriction of C^j to P are the same (namely $\bar{W}_0^j$) when restricted to "diagonal" currents, since an N-invariant on P can uniquely be reconstructed from its Weyl-invariant restriction to the space of diagonal currents. (The uniqueness of this reconstruction can be shown by an argument similar to the one that was used in the case of the Chevalley theorem.)

It is not hard to see that the Casimirs $\{C^j\}$ corresponding to the leading terms of a $\mathscr{W}$-basis $\{W^j\}$ form a basis in the ring of group-invariant polynomials. (It is enough to prove this for a $\mathscr{W}$-basis constructed by means of some DS gauge, but in this case these l Casimirs are independent even if restricted to the gauge section M_V, where they simply coincide with the l DS currents $\{U_j\}$.) So we can associate a Casimir basis to any $\mathscr{W}$-basis. On the other hand, it is also possible to choose some convenient basis for the Casimirs first, and then construct a $\mathscr{W}$-basis in such a way that the leading terms of the $\mathscr{W}$-generators are the given set of Casimirs. For example, for the case of A_l we can choose the Casimirs as

$$C^j = \frac{1}{j+1} \operatorname{Tr} J^{j+1}, \qquad j = 1, 2, ..., l. \tag{4.3}$$

Then we can define $\mathscr{W}$-generators corresponding to these Casimirs by the formula

$$W^j = \frac{1}{j+1} \operatorname{Tr} \hat{J}^{j+1},$$

where

$$\hat{J} = AJA^{-1} + \kappa A'A^{-1}$$

is the representative of the gauge orbit of the constrained current $J \in P$ in some particular DS gauge. (Remember that both $\hat{J} \in M_V$ and the gauge transformation A are uniquely determined by $J \in P$.) It follows that we have

$$W^j = \frac{1}{j+1} \operatorname{Tr} \hat{J}^{j+1} = \frac{1}{j+1} \operatorname{Tr}(J + \kappa A^{-1}A')^{j+1} = \frac{1}{j+1} \operatorname{Tr} J^{j+1} + \cdots, \tag{4.4}$$

that is the leading terms of the W^j are indeed the Casimirs C^j. It is also easy to see that the $\{W^j\}$ associated by this method to a set of independent Casimirs form a basis of $\mathscr{W}$-algebra. (The $\mathscr{W}$-generators associated to a given Casimir by means of different DS gauges differ in their derivative, non-leading terms.)

In the $SL(3, R)$ example, choosing the "Wronskian" gauge

$$\hat{J} = \begin{pmatrix} 0 & W^1 & W^2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

we have

$$W^1 = \tfrac{1}{2}\operatorname{Tr}\hat{J}^2 \qquad \text{and} \qquad W^2 = \tfrac{1}{3}\operatorname{Tr}\hat{J}^3. \tag{4.5}$$

By using the results of Subsection III.3 on the A_2 example we can derive the relations

$$\begin{aligned}
\{W^1(x), W^1(y)\} &= \kappa(W^1)'(x)\,\delta + 2\kappa W^1(x)\,\delta' - 2\kappa^3\,\delta''' \\
\{W^1(x), W^2(y)\} &= 2\kappa(W^2)'(x)\,\delta + 3\kappa W^2(x)\,\delta' \\
&\quad - \kappa^2[W^1(x)\,\delta]'' + \kappa^4\,\delta'''' \\
\{W^2(x), W^2(y)\} &= \kappa[\tfrac{2}{3}(W^1)'\,W^1 + \kappa(W^2)'' - \tfrac{2}{3}\kappa^2(W^1)'''](x)\,\delta \\
&\quad + \kappa[\tfrac{2}{3}(W^1)^2 + 2\kappa(W^2)' - 2\kappa^2(W^1)''](x)\,\delta' \\
&\quad - 2\kappa^3(W^1)'(x)\,\delta'' - \tfrac{4}{3}\kappa^3 W^1(x)\,\delta''' + \tfrac{2}{3}\kappa^5\,\delta'''''
\end{aligned} \tag{4.6}$$

where $\delta = \delta(x^1 - y^1)$ and $x^0 = y^0$. On the other hand, it is not difficult to verify that the corresponding Casimirs

$$C^1 = \tfrac{1}{2}\operatorname{Tr}J^2 \qquad \text{and} \qquad C^2 = \tfrac{1}{3}\operatorname{Tr}J^3 \tag{4.7}$$

satisfy the following algebra under Poisson bracket,

$$\begin{aligned}
\{C^1(x), C^1(y)\} &= \kappa(C^1)'(x)\,\delta + 2\kappa C^1(x)\,\delta' \\
\{C^1(x), C^2(y)\} &= 2\kappa(C^2)'(x)\,\delta + 3\kappa C^2(x)\,\delta' \\
\{C^2(x), C^2(y)\} &= \kappa[\tfrac{2}{3}(C^1)'\,C^1](x)\,\delta + \kappa[\tfrac{2}{3}(C^1)^2](x)\,\delta',
\end{aligned} \tag{4.8}$$

which is nothing but the leading term (in κ) of the full $\mathscr{W}$-algebra for $SL(3, R)$.

In fact we will show that in general, if the $\mathscr{W}$-generators W^i and W^j satisfy

$$\{W^i(x), W^j(y)\} = \sum_A f^A(W)(x)\,\delta^{(A)}(x^1 - y^1),$$

where the "structure functions" $f^A(W)$ are differential polynomials in $\{W^j\}$, then the corresponding Casimirs C^i and C^j satisfy the simplified (truncated) algebra

$$\{C^i(x), C^j(y)\} = f_0^1(C)(x)\,\delta' + f_1^0(C)(x)\,\delta, \tag{4.9a}$$

where f_0^1 and f_1^0 are the leading terms of f^1 and f^0 in the number of derivatives, which are 0 and 1, respectively.

To show this, let us first note that from the form of the KM Poisson brackets and group-invariance of the Casimirs it already follows that the commutator (4.9a) must be of the form

$$\{C^i(x), C^j(y)\} = g_0^1(J)(x)\,\delta' + g_1^0(J, J')(x)\,\delta \tag{4.9b}$$

with some group-invariant functions g_0^1 and g_1^0. (g_1^0 is polynomial in J, but is linear in J'.) Now we have to demonstrate that

$$g_0^1(J)=f_0^1(C(J)) \quad \text{and} \quad g_1^0(J,J')=f_1^0(C(J)). \tag{4.10}$$

We will make use of the diagonal gauge and the Chevalley theorem once more. In the diagonal gauge the leading terms of W^i and C^i coincide and therefore we have

$$\bar{g}_0^1(H)=f_0^1(\bar{C}(H)) \quad \text{and} \quad \bar{g}_1^0(H,H')=f_1^0(\bar{C}(H)). \tag{4.11}$$

Now applying the Chevalley theorem to g_0^1, the first equation in (4.10) follows from the first one in (4.11). Before one is able to apply the theorem also to g_1^0, one first has to generalize it for the case of operators containing one derivative. This is possible and the proof is basically the same as for operators without any derivatives. Let us define the group invariant $\delta_1^0(J,J')$ by

$$g_1^0(J,J')=f_1^0(C(J))+\delta_1^0(J,J').$$

From the second equation in (4.11) we see that $\bar{\delta}_1^0(H,H')$ vanishes, but then the full $\delta_1^0(J,J)$ must vanish too since

$$\delta_1^0(J,J')=\delta_1^0(H(J),(J')^g)=\delta_1^0(H(J),(H')^g)=\bar{\delta}_1^0(H(J),(H')^g)=0,$$

where the second step follows from the neutrality of the group-invariant δ_1^0.

This way we have shown that the set of Casimirs closes to form a polynomial algebra under the Poisson bracket and that this algebra is a truncated version of the corresponding $\mathscr{W}$-algebra. Since the completely local Casimirs $\{C^i\}$ are more elementary objects than the $\{W^i\}$ which contain derivatives as well, one can ask wether the closure of the Casimir algebra can be shown without any reference to the more complicated $\mathscr{W}$-algebra. In other words, one has to show that (4.9a) holds with some functions f_0^1 and f_1^0. It is trivial that g_0^1 in (4.9b) depends on J only through the Casimirs, since this merely expresses the fact that the $\{C^i\}$ form a basis for the completely local group-invariants.

To show that g_1^0 is also a function of the Casimirs we go to the diagonal gauge again. In this gauge the restriction of g_1^0 must be of the form

$$\bar{g}_1^0(H,H')=\sum_{i=1}^{l} A_i\theta_i', \tag{4.12}$$

where the $\{\theta_i\}$ are coordinates with respect to some basis in the Cartan subalgebra and the coefficients $\{A_i\}$ can be considered as an l-component vector in the Cartan subalgebra and can be expanded as

$$A_i=\sum_i B_j\frac{\partial\bar{C}^j(H)}{\partial\theta_i} \tag{4.13}$$

simply because the l vectors $\{\partial\bar{C}^j/\partial\theta_i\}$ are linearly independent. (This is the analytic expression of the fact that the l invariants $\{\bar{C}^j\}$ are functionally independent.)

Substituting (4.13) into (4.12) we find

$$\bar{g}_1^0(H, H') = \sum_j B_j \frac{\partial\bar{C}^j(H)}{\partial\theta_i}\theta_i' = \sum_j \frac{1}{h_j+1} B_j[\bar{C}^j(H)]'$$

and we see that the coefficients B_j must be Weyl-invariants:

$$\bar{g}_1^0(H, H') = \sum_j \frac{1}{h_j+1} B_j(\bar{C}(H))[\bar{C}^j(H)]'.$$

Now using the generalized Chevalley theorem for g_1^0 again we have

$$g_1^0(J, J') = \sum_j \frac{1}{h_j+1} B_j(C(J))[C^j(J)]'.$$

After this digression we return to the question of the quadratic closure of the $\mathcal{W}$-algebra. We have shown in the previous section that the $\mathcal{W}$-algebras for A_l, B_l, and C_l are quadratic in a suitably chosen basis. As an application of the relation between the $\mathcal{W}$-algebras and the algebras of the corresponding Casimirs we now prove that no such basis exists for D_l and the exceptional algebras. In fact we show this for the Casimir algebras, from which the analogous result for the $\mathcal{W}$-algebras immediately follows.

Let C_H be the highest order Casimir, of order H (see Appendix A), and let us consider the Poisson bracket of C_H with itself,

$$\{C_H(x), C_H(y)\} = \Gamma_{2H-2}(C)(x)\,\delta' + \tfrac{1}{2}\Gamma'_{2H-2}(C)(x)\,\delta. \tag{4.14}$$

(Here the two structure functions are not independent of each other due to the antisymmetry of the Poisson bracket.) The structure function $\Gamma_{2H-2}(C)$ is a Casimir of order $(2H-2)$ and by inspecting the list of group-invariants for the case of the exceptional groups it is easy to see that it can never be expressed as a quadratic function of the basic Casimirs $\{C^j\}$ for these groups.

The situation for D_l is more complicated. Here we can show that the set of Casimirs $\{C^1, C^2, ..., C^l\}$ defined by

$$\det(1-\sqrt{\mu}\,J) = 1 - \sum_{n=1}^{l} \mu^n C^n, \tag{4.15}$$

where the determinant is taken in the $2l$-dimensional vector representation of D_l, forms a quadratic algebra under Poisson bracket. (This is actually the same algebra as is formed by the corresponding Casimirs of the B_l and C_l groups.) However, as is well known, (4.15) is not a correct choice of basic Casimirs for D_l, the latter is given by the set $\{C^1, C^2, ..., C^{l-1};\ C_\sigma = \sqrt{C^l}\}$. By introducing the "spinorial" invariant C_σ, we destroy the quadratic property of the algebra. We find that Γ_{4l-6}, the structure function in the commutator of two highest Casimirs C^{l-1} is given by (see Appendix C)

$$\Gamma_{4l-6} = -12\kappa(C_\sigma)^2\, C^{l-3} - 4\kappa C^{l-1} C^{l-2} \tag{4.16}$$

which is indeed cubic for $l > 3$.

IV.2. *Explicit Casimir Algebras*

In Subsection IV.1 we have shown that the Casimir operators, C^n, form a closed, polynomial algebra under Poisson bracket, which is a truncated version of the $\mathcal{W}$-algebra. These Casimir algebras are interesting in their own right and they are also useful for studying the related $\mathcal{W}$-algebras. In this subsection we exhibit their structure in some detail.

First, it is obvious that the Casimirs are conformal primary fields with respect to the Sugawara energy–momentum tensor. Next we want to determine the non-trivial Poisson bracket relations describing this algebra. What we are actually going to calculate is the Poisson bracket of the generating polynomials

$$A(\mu, x) = \det(1 - \mu J(x)) = 1 - \sum_{n=1}^{l} \mu^{n+1} C^n(x) \tag{4.17a}$$

and

$$B(\mu, x) = \det(1 - \sqrt{\mu}\, J(x)) = 1 - \sum_{n=1}^{l} \mu^n C^n(x) \tag{4.17b}$$

for the l independent Casimirs $C^1, ..., C^l$ of the A_l and B_l (C_l) algebras, respectively. One first observes that the overcomplete set of group-invariant polynomials

$$Q^n(x) = \frac{1}{n} \operatorname{Tr} J^n(x), \qquad n = 2, 3, ... \tag{4.18}$$

which are related to the l independent Casimirs C^n via

$$C^{n-1} = -\frac{1}{n!} \frac{d^n}{d\mu^n} \exp\left(-\sum_{r=2}^{\infty} \mu^r Q^r\right)\bigg|_{\mu=0}, \qquad \text{for } A_l,$$

$$C^n = -\frac{1}{n!} \frac{d^n}{d\mu^n} \exp\left(-2\sum_{r=1}^{\infty} \mu^r Q^{2r}\right)\bigg|_{\mu=0}, \qquad \text{for } B_l, C_l,$$

satisfy the Poisson brackets (see Appendix C)

$$\{Q^n(x), Q^m(y)\} = \kappa\left[(p-2)Q^{p-2} - \frac{q}{N}Q^{n-1}Q^{m-1}\right](x)\,\delta' + \kappa\left[(m-1)(Q^{p-2})' - \frac{q}{N}Q^{n-1}(Q^{m-1})'\right](x)\,\delta, \quad (4.19)$$

where δ stands for $\delta(x^1 - y^1)$ as before, $p = n + m$, $q = (n-1)(m-1)$, and N is the dimension of the defining representation. Note in particular that for the B and C algebras both n and m must be even integers (since for odd n the Q^n vanish identically) and, as a consequence, the quadratic terms on the right hand side of (4.19) are automatically absent.

However, formula (4.19) is only the first step in finding the explicit Casimir algebras. For example, in the case of A_2 one obtains

$$\{Q^3(x), Q^3(y)\} = \kappa\left(4Q^4 - \frac{4}{3}Q^2Q^2\right)\delta' + \frac{\kappa}{2}\left(4Q^4 - \frac{4}{3}Q^2Q^2\right)'\delta$$

and only after expressing Q^4 in terms of the independent Casimirs Q^2 and Q^3 via $2Q^4 = (Q^2)^2$ does one find the result (4.8) (note that $Q^2 = C^1$ and $Q^3 = C^2$ there). More generally, if one computes the Poisson brackets of the highest Casimirs for an algebra of rank l, one has to use the characteristic polynomial $O(l/2)$ times to express the right hand side of (4.19) in terms of the independent Casimirs. Clearly this method becomes soon cumbersome and another algorithm is needed.

As a first step to calculate the Poisson bracket of the generating polynomial (with itself) we expand its logarithm,

$$\log\det(1 - \mu J(x)) = -\sum_{n=2}^{\infty} \mu^n Q^n(x) \quad (4.20)$$

and use (4.19) to calculate the Poisson bracket of $\log\det(1 - \mu J)$. This then allows for the computation of the Poisson brackets of the determinant. After some algebra one finds that this Poisson bracket can be reexpressed in terms of the determinant and its derivatives. For the details of the derivation we refer the reader to Appendix C. The final results are

A_l algebras:

$$\{A(\mu, x), A(\nu, y)\} = \kappa\mu^2\nu^2\left(\frac{1}{\mu-\nu}(\partial_\nu - \partial_\mu) - \frac{1}{l+1}\partial_\mu\partial_\nu\right)A(\mu, x)\,A(\nu, x)\,\delta' + \kappa\mu^2\nu^2\left((\partial_\nu - \partial_\mu)\frac{1}{\mu-\nu} - \frac{1}{l+1}\partial_\mu\partial_\nu\right)A(\mu, x)\,\partial_x A(\nu, x)\,\delta + \frac{\kappa\mu^2\nu^2}{(\mu-\nu)^2}(A(\mu, x)\,\partial_x A(\nu, x) - A(\nu, x)\,\partial_x A(\mu, x))\,\delta, \quad (4.21)$$

B_l, C_l algebras:

$$\{B(\mu, x), B(\nu, y)\} = 4\kappa\mu\nu \frac{1}{\mu-\nu} (\nu\partial_\nu - \mu\partial_\mu) B(\mu, x) B(\nu, x) \delta' + 4\kappa\mu\nu(\nu\partial_\nu - \mu\partial_\mu) \frac{1}{\mu-\nu} (B(\mu, x) \partial_x B(\nu, x)) \delta + 2\kappa\mu\nu \frac{\mu+\nu}{(\mu-\nu)^2} (B(\mu, x) \partial_x B(\nu, x) - B(\nu, x) \partial_x B(\mu, x)) \delta. \tag{4.22}$$

The algebra of the Casimirs can now be computed by inserting the expansions (4.17) into both sides of (4.21), resp. (4.22), and comparing coefficients in the resulting polynomials in μ and ν. One sees, in particular, that with respect to the Casimirs defined by the determinant the algebras close always quadratically.

For example, for the highest Casimirs of A_l, $l \geqslant 2$ one obtains

$$\{C^l(x), C^l(y)\} = \kappa a_l(x) \delta' + \frac{\kappa}{2} a_l'(x) \delta$$
$$a_l = -2C^l C^{l-2} \theta(l-3) + \frac{l}{l+1} (C^{l-1})^2 \tag{4.23}$$

and for the highest Casimirs of B_l and C_l with $l \geqslant 2$ one finds

$$\{C^l(x), C^l(y)\} = -4\kappa C^l C^{l-1} \delta' - 2\kappa (C^l C^{l-1})' \delta. \tag{4.24}$$

The corresponding results for the lower Casimirs are presented in Appendix C.

V. Differential and Pseudo-Differential Operators and Toda Fields

The aim of this last section is to demonstrate that the differential and pseudo-differential operators studied in [10], and taken as a starting point for the quantization of $\mathscr{W}$-algebra in [5], arise naturally in our framework. These operators appear in the differential equations satisfied by the gauge-invariant components of the WZNW field.

Let us recall that the solution of the field equations for the group-valued field g is

$$g(x^+, x^-) = g_L(x^+) \cdot g_R(x^-) \tag{5.1}$$

with

$$g_L' g_L^{-1} = J \qquad \text{and} \qquad g_R^{-1} g_R' = \tilde{J}, \tag{5.2}$$

where the currents J and $\tilde{J}$ are subject to the constraints (2.8). (In this section prime means $2\kappa\, \partial/\partial x^1$.) We will consider the simplest case, $SL(n, R)$ first, and concentrate on the left-moving part of the theory. (We omit the subscript L.)

In order to reconstruct the group-valued field $g(x^+)$ from the current $J(x^+)$ (satisfying the constraints (2.8)), one has to solve the set of linear differential equations

$$g' = Jg. \tag{5.3}$$

Obviously this is a separate set of equations for each column-vector of the matrix g, which are of the form

$$\begin{pmatrix} g'_{1i} \\ g'_{2i} \\ \vdots \\ g'_{ni} \end{pmatrix} = (I_- + j) \begin{pmatrix} g_{1i} \\ g_{2i} \\ \vdots \\ g_{ni} \end{pmatrix}. \tag{5.4}$$

Solving (5.4) is the simplest in the "Wronskian" gauge, (3.71). In this gauge one can easily express all components of g in terms of the bottom components, g_{ni}, denoted by ψ_i,

$$\begin{aligned} g_{(n-1)i} &= \psi'_i \\ g_{(n-2)i} &= \psi''_i \\ &\vdots \\ g_{1i} &= \psi_i^{(n-1)} \end{aligned} \tag{5.5}$$

leading to a single nth-order differential equation satisfied by ψ_i,

$$\psi_i^{(n)} = \sum_{j=1}^{n-1} U^j \psi_i^{(n-j-1)}. \tag{5.6}$$

The group-valued field g can now be built from the n independent solutions of (5.6),

$$g = \begin{pmatrix} \psi_1^{(n-1)} & \psi_2^{(n-1)} & \cdots & \psi_n^{(n-1)} \\ \vdots & \vdots & & \vdots \\ \psi'_1 & \psi'_2 & \cdots & \psi'_n \\ \psi_1 & \psi_2 & \cdots & \psi_n \end{pmatrix}, \tag{5.7}$$

where the set of solutions $\{\psi_i\}$ must satisfy the Wronskian constraint (hence the name of the gauge):

$$\det g = 1 \tag{5.8}$$

in order that the matrix g be an element of the group $SL(n, R)$.

We note that if the DS currents $\{U^j\}$ are regular functions then so are the solutions $\{\psi_i\}$ of the generalized Schrödinger equation (5.6). By combining g_L given by (5.7) with the similarly constructed right-moving solution g_R, the resulting WZNW field $g(x^+, x^-)$ is also regular, as are the globally defined Toda fields, being subdeterminants of the latter (according to (2.23)). Furthermore, if the $\mathscr{W}$-generators corresponding to a given (say, the "Wronskian") DS gauge are given by regular functions for a Toda solution, then by this procedure one can always construct a regular WZNW representative of that Toda solution, whether or not the solution appears to be regular in terms of the traditional local Toda variables, ϕ^α.

We also remark that once the solutions of the "right handed" analogue of (5.6), $\{\chi_i\}$, are known, then as a consequence of (2.24) and (5.1), the Toda fields can be expressed in terms of the $\{\psi\}$'s and the $\{\chi\}$'s as

$$\begin{aligned}
e^{-\phi_{n-1/2}} &= \mathscr{D}_n = \psi \cdot \chi \\
e^{-\phi_{n-2/2}} &= \mathscr{D}_{n-1} = \det \begin{pmatrix} \psi' \cdot \chi' & \psi' \cdot \chi \\ \psi \cdot \chi' & \psi \cdot \chi \end{pmatrix} \\
&= (\psi' \cdot \chi')(\psi \cdot \chi) - (\psi' \cdot \chi)(\psi \cdot \chi') \\
e^{-\phi_{n-3/2}} &= \mathscr{D}_{n-2} = \det \begin{pmatrix} \psi'' \cdot \chi'' & \psi'' \cdot \chi' & \psi'' \cdot \chi \\ \psi' \cdot \chi'' & \psi' \cdot \chi' & \psi' \cdot \chi \\ \psi \cdot \chi'' & \psi \cdot \chi' & \psi \cdot \chi \end{pmatrix} \\
&= (\psi'' \cdot \chi'')(\psi' \cdot \chi')(\psi \cdot \chi) + \cdots
\end{aligned} \tag{5.9}$$

where

$$\psi \cdot \chi = \sum_i \psi_i \cdot \chi_i, \qquad \psi' \cdot \chi = \sum_i \psi_i' \cdot \chi_i, \tag{5.10}$$

and so on. In fact Eq. (5.9) was the starting point of the analysis of Toda theory in [8]. Without going into details we note that the above results can easily be generalized for the B_l and C_l series.

So far we have studied (5.4) in a definite DS gauge. Let us now try to solve it for g without gauge-fixing the current J. It is easy to see that starting from the bottom row, it is always possible to eliminate all higher components of g successively, even without any gauge fixing. This elimination leads to a differential equation of the form

$$\mathscr{D}_n^{(A)} \psi_i = \partial^n \psi_i - \sum_{j=1}^{n-1} W^j[J(x^+)]\, \partial^{n-j-1} \psi_i = 0. \tag{5.11}$$

Here $\partial = \kappa\, \partial/\partial x^+$ and the coefficient functions $\{W^j\}$ are automatically obtained as some differential polynomials in the current components. Moreover, they are gauge invariant, since the original equation (5.3) was gauge-covariant and the bottom components $g_{ni} = \psi_i$ are gauge-invariant (with respect to left-moving upper triangular gauge transformations). This implies that the W^j's in (5.11) are nothing but the $\mathscr{W}$-generators associated to the "Wronskian" gauge, since they reduce to the DS currents $\{U^j\}$ in this gauge.

To summarize, if the $\mathscr{W}$-densities associated to a DS gauge (here the "Wronskian" gauge) are known, then one can reconstruct the corresponding WZNW solution by solving (5.2) for $g = g_L \cdot g_R$ in that DS gauge. In the reconstruction procedure one obtains a higher order differential equation (here (5.6)) satisfied by the gauge-invariant (bottom row) components of g_L (and an analogous equation for the last column of g_R). The same equation can also be derived from (5.4) by elimination without any gauge fixing. Since the resulting differential equation is gauge-invariant, one can read off the explicit formula of the $\mathscr{W}$-generators corresponding to the given DS gauge by comparing the coefficients in the differential equations obtained with and without gauge-fixing. By a similar argument, one can also establish the transformation rules relating the $\mathscr{W}$-bases corresponding to different DS gauges.

The elimination is also simple in the diagonal gauge. In this gauge

$$j = \begin{pmatrix} \theta_1 & 0 & \cdots & 0 \\ 0 & \theta_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \theta_n \end{pmatrix}, \tag{5.12}$$

and the differential operator takes the form

$$\mathscr{D}_n^{(A)} = (\partial - \theta_1)(\partial - \theta_2) \cdots (\partial - \theta_n). \tag{5.13}$$

By rearranging this product as a sum corresponding to (5.11), one can read off the expression of the $\mathscr{W}$-generators in this gauge. Note that the diagonal fields (5.12) are not independent, because $\theta_1 + \theta_2 + \cdots + \theta_n = 0$. This is the original form of the Miura-transformation [21] and the operator (5.13) is the starting point for the Lukyanov–Fateev free-field construction of quantized $\mathscr{W}$-algebra [5].

The derivation of the gauge-invariant higher order differential operators and the reconstruction of the matrix-valued field g from the constrained currents (or from $\mathscr{W}$-generators) proceeds analogously for the Lie algebras B_n and C_n. The resulting gauge-invariant differential operators are of order $(2n+1)$ and $(2n)$, respectively, according to the dimensions of the defining representations. Due to the restrictions (3.86) and (3.82), the differential operators $\mathscr{D}_n^{(B)}$ and $\mathscr{D}_n^{(C)}$ are (formally) antiselfadjoint and selfadjoint, respectively. Without going into detail, we give these operators in the factorized form corresponding to the diagonal gauge:

$$\mathscr{D}_n^{(B)} = (\partial - \theta_1)(\partial - \theta_2) \cdots (\partial - \theta_n)\, \partial (\partial + \theta_n) \cdots (\partial + \theta_2)(\partial + \theta_1) \tag{5.14a}$$

$$\mathscr{D}_n^{(C)} = (\partial - \theta_1)(\partial - \theta_2) \cdots (\partial - \theta_n)(\partial + \theta_n) \cdots (\partial + \theta_2)(\partial + \theta_1). \tag{5.14b}$$

Here the θ_i's are independent free fields.

The case of the algebras D_n is more complicated. As an example, let us consider D_3 first. We use the six dimensional vector representation and go to the diagonal gauge, where (with a convenient choice of the τ_i)

$$J = I_- + j = \begin{pmatrix} \theta_1 & 0 & 0 & 0 & 0 & 0 \\ 1 & \theta_2 & 0 & 0 & 0 & 0 \\ 0 & 1 & \theta_3 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\theta_3 & 0 & 0 \\ 0 & 0 & -1 & -1 & -\theta_2 & 0 \\ 0 & 0 & 0 & 0 & -1 & -\theta_1 \end{pmatrix}. \tag{5.15}$$

If we write out (5.4) in components we have (suppressing the index i)

$$g_5 = -(\partial + \theta_1)\, g_6 \tag{5.16a}$$

$$g_3 + g_4 = -(\partial + \theta_2)\, g_5 \tag{5.16b}$$

$$g_2 = (\partial + \theta_3)\, g_4 \tag{5.16c}$$

$$g_2 = (\partial - \theta_3)\, g_3 \tag{5.16d}$$

$$g_1 = (\partial - \theta_2)\, g_2 \tag{5.16e}$$

$$0 = (\partial - \theta_1)\, g_1. \tag{5.16f}$$

From (5.16) we see that the elimination is blocked here after the second step, since the combination $g_3 - g_4$ never occurs on the left hand side. On the other hand, its derivative can be expressed, combining (5.16c) and (5.16d), as

$$\partial(g_3 - g_4) = \theta_3(g_3 + g_4). \tag{5.17}$$

One can go on with the elimination by integrating (5.17) (formally) using the "antiderivation" symbol ∂^{-1}:

$$(g_3 - g_4) = \partial^{-1}\theta_3(g_3 + g_4). \tag{5.18}$$

One then finds

$$\begin{aligned} \mathscr{D}_3^{(D)} &= (\partial - \theta_1)(\partial - \theta_2)(\partial - \theta_3\,\partial^{-1}\theta_3)(\partial + \theta_2)(\partial + \theta_1) \\ &= (\partial - \theta_1)(\partial - \theta_2)(\partial - \theta_3)\,\partial^{-1}(\partial + \theta_3)(\partial + \theta_2)(\partial + \theta_1). \end{aligned} \tag{5.19}$$

Similarly, by performing the elimination in the $(2n)$-dimensional vector representation, one can associate a pseudo-differential operator to any D_n algebra:

$$\mathscr{D}_n^{(D)} = (\partial - \theta_1)(\partial - \theta_2)\cdots(\partial - \theta_n)\,\partial^{-1}(\partial + \theta_n)\cdots(\partial + \theta_2)(\partial + \theta_1). \tag{5.20}$$

This not only shows that it is impossible to obtain a *differential* operator for D_n in the vector representation, but from the example of $D_3 \sim A_3$ we also see that the type of pseudo-differential operator depends on the representation in which (5.4) is taken. (For D_3 there is an ordinary differential operator corresponding to the four dimensional representation, but this is the spinor of D_3.)

For the case of A_n, B_n, and C_n what makes the elimination simple is that the matrix I_- (see (3.70) and (3.83)) has non-zero entries immediately below the diagonal and only there. Since I_- is the negative step-operator of the special $sl(2, R)$ subalgebra $\mathscr{S}$ introduced in Subsection III.1, this fact means that the defining representations of these algebras are still irreducible with respect to this $sl(2, R)$ subalgebra.

For D_n, the vector representation is reducible with respect to $\mathscr{S}$ with branching $2n = (2n-1) + 1$. This is why one has a pseudo-differential, rather than a differential, operator after eliminating the higher components from the system of first order differential equations (5.4). (The spinor representations of D_n are even worse from this point of view, except for $n = 3$.)

Turning to the exceptional algebras, we find that the seven dimensional representation of G_2 is irreducible with respect to $\mathscr{S}$ and therefore the elimination for G_2 will result in a 7th order differential operator (see Appendix B). The corresponding branching rule for F_4 is [22] $26 = 17 + 9$, so in this case we have a pseudo-differential operator, containing one integration.

Finally, for E_6, E_7, and E_8 the branching rules are [22]

$$
\begin{aligned}
&E_6:\ 27 = 17 + 9 + 1\\
&E_7:\ 56 = 28 + 18 + 10\\
&E_8: 248 = 59 + 47 + 39 + 35 + 27 + 23 + 15 + 3
\end{aligned}
\tag{5.21}
$$

and therefore in these cases the elimination leads to pseudo-differential operators, containing 2, 2, and 7 integrations, respectively.

VI. Conclusions

In this paper we have shown that extended conformal algebras, $\mathscr{W}$-algebras arise naturally in the constrained WZNW formulation of Toda field theories. Our main results are the following:

We have given an ambidextrous generalization of the usual gauged WZNW models to derive Toda theories. Using the embedding WZNW phase space, we have shown how to implement the action of the $\mathscr{W}$-algebra generators as certain field dependent Kac–Moody transformations. This led us to a powerful algorithm to calculate the $\mathscr{W}$-algebra relations. Using this algorithm we calculated the so far unknown Poisson bracket algebra of $\mathscr{W}(G_2)$ explicitly.

We exhibited a particular basis where all the $\mathscr{W}$-generators are conformal primary fields. We have also shown that for the A, B, C series there is always a basis in which the $\mathscr{W}$-algebra closes quadratically, and that is not true for the rest of the simple Lie algebras. Finally we have proved that the leading terms of the $\mathscr{W}$-generators (i.e., terms without derivatives) are restricted Casimir operators. We exhibited the Casimir algebra relations in detail for the A, B, and C series and have given a general proof of closure of their Poisson bracket algebra for any simple Lie algebra.

As found in [15] the quantum version of the Casimir algebra does not close in general (it has been shown to close for $SU(3)$ only when the level is equal to one) hence it is natural to ask whether an extension of the $\mathscr{W}$-generators to the full Kac–Moody phase space (in the sense discussed in the introduction of Section III) exists, with the full, unrestricted Casimirs as leading terms. If one could find such an extension it would make it possible to associate a representation of the $\mathscr{W}$-algebra in terms of unrestricted Kac–Moody generators to any Kac–Moody algebra. As such an extension would also be a deformation of the Casimir algebra, it could possibly survive quantization. This problem is certainly very interesting as it would also clarify the origin of $\mathscr{W}$-algebras, without making reference to any particular model. A detailed investigation of such deformations of the Casimir algebras is outside the scope of this paper. After some preliminary investigation of the problem we found that at least for A_2 one cannot extend the $\mathscr{W}$-algebra to the whole phase space of the (chiral) Kac–Moody currents, at least with the assumption of the W^i's being differential polynomials with unrestricted Casimirs as leading terms.

However, by giving up the polynomial nature of the W^i's we have found such an extension of the generators of the $\mathscr{W}$-algebra for A_2. In fact this result can be generalized for an arbitrary A_n algebra. This problem is under investigation.

Appendix A: Conventions

Here we give our conventions and present some formulae which are used in the paper.

Space–Time and Poisson Brackets.

$$\eta_{00} = -\eta_{11} = 1, \qquad x^{\pm} = \tfrac{1}{2}(x^0 \pm x^1), \qquad \partial_{\pm} = \partial_0 \pm \partial_1. \tag{A.1}$$

We use equal time Poisson brackets and spatial δ-distributions. At fixed x^0 all quantities are supposed to be periodic with period 2π. Prime means "twice spatial derivative" everywhere, even on Dirac δ's. Note that this is equal to ∂_+ on quantities depending on x through x^+ only.

Conformal Primary Fields. The left-moving conformal transformations are generated by the conserved moments

$$Q_a = \int_0^{2\pi} dx^1\, a(x)\, L(x) \tag{A.2}$$

of the Virasoro density $L(x) = \Theta_{++}(x)$, for any periodic test function $a(x)$ for which $\partial_- a(x)=0$. A conformal primary field Ψ of left conformal weight Δ transforms as

$$(\delta_L \Psi)(y) = -\{Q_a, \Psi(y)\} = a(y)\,\partial_+ \Psi(y) + \Delta \cdot \Psi(y)\,\partial_+ a(y). \tag{A.3}$$

If $\partial_- \Psi = 0$ then this is equivalent to

$$\{L(x), \Psi(y)\}_{|x^0=y^0} = \Delta \cdot \Psi(x)\,\delta'(x^1-y^1) + (\Delta - 1)\cdot(\partial_+ \Psi(x))\,\delta(x^1-y^1). \tag{A.4}$$

Lie Algebras. Let $\mathscr{G}_c$ be a complex simple Lie algebra, Φ the set of roots with respect to some Cartan subalgebra, and Δ a set of simple roots. There is a Cartan element H_φ associated to every $\varphi \in \Phi$ and the Cartan matrix $K_{\alpha\beta}$ is given as

$$K_{\alpha\beta} = \alpha(H_\beta) = \frac{2\alpha\cdot\beta}{|\beta|^2} = \frac{|\alpha|^2}{2}\,\mathrm{Tr}(H_\alpha \cdot H_\beta), \qquad \alpha, \beta \in \Delta, \tag{A.5}$$

where Tr is the usual matrix trace multiplied by an appropriate normalization constant, which ensures that $|\alpha_{\mathrm{long}}|^2 = 2$. For example, for the defining representations of the orthogonal Lie algebras B_l and D_l this normalization constant is $\frac{1}{2}$, and it is 1 for the defining representations of A_l and C_l. For any positive root $\alpha \in \Phi^+$ we choose step operators $E_{\pm\alpha}$ so that we have

$$H_\alpha = [E_\alpha, E_{-\alpha}], \qquad \mathrm{Tr}(E_\alpha \cdot E_\beta) = \frac{2}{|\alpha|^2}\,\delta_{\alpha,-\beta}, \mathrm{Tr}(E_\alpha \cdot H_\beta) = 0 \tag{A.6}$$

for $\alpha, \beta \in \Phi$, and also

$$[H_\alpha, E_\beta] = K_{\beta\alpha} E_\beta, \qquad \text{for} \quad \alpha, \beta \in \Delta. \tag{A.7}$$

In our Cartan–Weyl basis H_α ($\alpha \in \Delta$), $E_{\pm\varphi}$ ($\varphi \in \Phi^+$) all the structure constants of $\mathscr{G}_c$ are real numbers. Throughout the paper we use the *maximally non-compact* real form $\mathscr{G}$ of $\mathscr{G}_c$ for which the Cartan decomposition is valid without complexification. We in fact take $\mathscr{G}$ to be the real span of the Cartan–Weyl basis of $\mathscr{G}_c$. The maximally non-compact real forms of A_l, B_l, C_l, and D_l are (up to isomorphism) $sl(l+1, R)$, $so(l, l+1, R)$, $sp(2l, R)$, *and* $so(l, l, R)$, respectively.

The exponents of the simple Lie algebras are listed in the following table:

Algebra	Exponents
A_l	$1, 2, \dots, l$
B_l	$1, 3, \dots, 2l-1$
C_l	$1, 3, \dots, 2l-1$
D_l	$1, 3, \dots, 2l-3; l-1$
G_2	1, 5
F_4	1, 5, 7, 11
E_6	1, 4, 5, 7, 8, 11
E_7	1, 5, 7, 9, 11, 13, 17
E_8	1, 7, 11, 13, 17, 19, 23, 29

Kac–Moody Algebras. We denote the space of $\mathcal{G}$-valued left-moving currents by K. The KM Poisson brackets of the components of $J(x) = J^a(x) T_a$ are given as

$$\{J^a(x), J^b(y)\}_{|x^0 = y^0} = f^{ab}_c J^c(x)\, \delta(x^1 - y^1) + \kappa g^{ab}\, \delta'(x^1 - y^1), \tag{A.8}$$

where the f^{ab}_c are the structure constants, the KM level k is $-4\pi\kappa$, and Lie algebra indices are raised and lowered by using the metric

$$g_{ab} = \mathrm{Tr}(T_a \cdot T_b). \tag{A.9}$$

APPENDIX B: G_2 $\mathcal{W}$-ALGEBRA

In this appendix, as a non-trivial example, we compute the Poisson-bracket relations of the $\mathcal{W}$-algebra corresponding to G_2 explicitly, using the tangential Kac–Moody method introduced in Subsection III.3.

We will work in the seven dimensional representation of G_2 and choose the following matrix representation for the two simple step operators:

$$E_\alpha = \begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&-1&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix}, \qquad E_\beta = \begin{pmatrix} 0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&\sqrt{2}&0&0&0\\ 0&0&0&0&-\sqrt{2}&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&-1\\ 0&0&0&0&0&0&0 \end{pmatrix}. \tag{B.1}$$

Choosing $\tau_1 = \tau_2 = 1$, the generators of the special $sl(2, R)$ subalgebra are

$$I_- = E^t_\alpha + E^t_\beta, \qquad I_+ = 10E_\alpha + 6E_\beta, \qquad \hat{\rho} = \tfrac{1}{2}[I_+, I_-], \tag{B.2}$$

where $\hat{\rho}$ is a diagonal matrix with diagonal elements 3, 2, 1, 0, -1, -2, -3, respectively.

Since there is no "quadratic" gauge for G_2, the only distinguished gauge is the "highest weight" DS gauge and we will work in this gauge. We denote the two DS fields by L and Z, which are the coefficients of $\frac{1}{2}I_+$ and the step operator for the highest root, respectively:

$$J = \begin{pmatrix} 0&3L&0&0&0&Z&0\\ 1&0&5L&0&0&0&-Z\\ 0&1&0&3\sqrt{2}L&0&0&0\\ 0&0&\sqrt{2}&0&-3\sqrt{2}L&0&0\\ 0&0&0&-\sqrt{2}&0&-5L&0\\ 0&0&0&0&-1&0&-3L\\ 0&0&0&0&0&-1&0 \end{pmatrix}. \tag{B.3}$$

On the other hand, a general Lie algebra element can be parametrized as

$$R=\begin{pmatrix} H_1 & A_1 & B & \sqrt{2}\,C & D & E & 0 \\ a_1 & H_2 & A_2 & -\sqrt{2}\,B & C & 0 & -E \\ b & a_2 & H_3 & \sqrt{2}\,A_1 & 0 & -C & -D \\ \sqrt{2}\,c & -\sqrt{2}\,b & \sqrt{2}\,a_1 & 0 & -\sqrt{2}\,A_1 & \sqrt{2}\,B & -\sqrt{2}\,C \\ d & c & 0 & -\sqrt{2}\,a_1 & -H_3 & -A_2 & -B \\ e & 0 & -c & \sqrt{2}\,b & -a_2 & -H_2 & -A_1 \\ 0 & -e & -d & -\sqrt{2}\,c & -b & -a_1 & -H_1 \end{pmatrix}, \tag{B.4}$$

where $H_1 = H_2 + H_3$.

Now we have to solve the equation

$$\delta J = [R, J] + \kappa R' \tag{B.5}$$

for the variations δL, δZ parametrizing δJ in terms of the independent parameters of R, which are the parameter e and a certain linear combination of a_1 and a_2, which correspond to the variations generated by Z and L, respectively.

Let us discuss the conformal variation first. Using (3.22b), we see that the properly normalized conformal generator is

$$\Lambda = \frac{14}{\kappa} L. \tag{B.6}$$

The conformal variations, generated by the conformal "charge"

$$Q_\alpha = \int_0^{2\pi} dx^1\, \alpha(x)\, \Lambda(x) \tag{B.7}$$

(through Dirac-brackets) are obtained by solving (B.5) with

$$e = 0 \qquad \text{and} \qquad a_1 = a_2 = \frac{1}{\kappa}\alpha. \tag{B.8}$$

We find

$$\begin{aligned} \{\Lambda, Q_\alpha\}^* &= \delta_\alpha \Lambda = \alpha\Lambda' + 2\alpha'\Lambda - 14\kappa\alpha''' \\ \{Z, Q_\alpha\}^* &= \delta_\alpha Z = \alpha Z' + 6\alpha' Z. \end{aligned} \tag{B.9}$$

From (B.9) we see that the central charge of the Virasoro algebra is

$$c = -168k \tag{B.10}$$

and that the field Z is a conformal primary field with conformal spin 6.

The only non-trivial $\mathcal{W}$-transformation is generated by

$$Q_e = \int_0^{2\pi} dx^1\, e(x)\, Z(x). \tag{B.11}$$

The corresponding variations can be found by solving (B.5) now with

$$e \neq 0 \qquad \text{and} \qquad 9a_1 + 5a_2 = 0. \tag{B.12}$$

After a lengthy computation we find

$$\{Z, Q_e\}^* = \delta_e Z$$
$$= \frac{1}{168}\left\{ -\kappa^{11} e^{(11)} + \sum_{i=0}^{4} \kappa^{2i+1} [(eQ_{2i+1})^{(2i+1)} + e^{(2i+1)} Q_{2i+1}] \right\}, \tag{B.13}$$

where

$$\begin{aligned}
Q_1 = & -4576L^2 Z - 756\kappa^2 L'' Z - 1850\kappa^2 L'Z' - 860\kappa^2 LZ'' - 74\kappa^4 Z''''\\
& + 230400L^5 + 407392\kappa^2 L^3 L'' + 1514056\kappa^2 L^2 (L')^2 + 111956\kappa^4 L^2 L''''\\
& + 1010254\kappa^4 LL'L''' + 797637\kappa^4 L(L'')^2 + 1648812\kappa^4 (L')^2 L'' + 21196\kappa^6 LL^{(6)}\\
& + 138201\kappa^6 L'L''''' + 364431\kappa^6 L''L'''' + \tfrac{495117}{2}\kappa^6 (L''')^2 + 2073\kappa^8 L^{(8)},\\
Q_3 = & \; 1240LZ + 120\kappa^2 Z'' - 168608L^4 - 184316\kappa^2 L^2 L'' - 457655\kappa^2 L(L')^2\\
& - 34870\kappa^4 LL'''' - 157520\kappa^4 L'L''' - 124443\kappa^4 (L'')^2 - 3410\kappa^6 L^{(6)},\\
Q_5 = & -52Z + 30580L^3 + 17226\kappa^2 LL'' + \tfrac{42867}{2}\kappa^2 (L')^2 + 1683\kappa^4 L'''',\\
Q_7 = & -2046L^2 - 396\kappa^2 L'',\\
Q_9 = & \; 55L.
\end{aligned}$$

Note that it is a non-trivial check on our result for $\delta_e Z$ that it can be written in the form (B.13), which follows from the antisymmetry of the $\{Z, Z\}^*$ Dirac-bracket hidden in $\{Z, Q_e\}^*$.

Finally, by introducing an orthonormal basis $\{H_1, H_2\}$ in the Cartan subalgebra defined by

$$[E_\alpha, E'_\alpha] = \sqrt{2}\, H_1, \qquad [E_\beta, E'_\beta] = -\frac{3}{\sqrt{2}} H_1 + \frac{3}{\sqrt{6}} H_2 \tag{B.14}$$

and going to the diagonal gauge where

$$J = I_- + \theta_1 H_1 + \theta_2 H_2 \tag{B.15}$$

we can easily write down the 7th-order differential operator discussed in Section V:

$$\mathscr{D}_2^{(G)}=\left(\kappa\partial-\frac{2}{\sqrt{6}}\theta_2\right)\left(\kappa\partial-\frac{1}{\sqrt{6}}\theta_2-\frac{1}{\sqrt{2}}\theta_1\right)\left(\kappa\partial-\frac{1}{\sqrt{6}}\theta_2+\frac{1}{\sqrt{2}}\theta_1\right)$$
$$\cdot(\kappa\partial)\left(\kappa\partial+\frac{1}{\sqrt{6}}\theta_2-\frac{1}{\sqrt{2}}\theta_1\right)\left(\kappa\partial+\frac{1}{\sqrt{6}}\theta_2+\frac{1}{\sqrt{2}}\theta_1\right)\left(\kappa\partial+\frac{2}{\sqrt{6}}\theta_2\right). \tag{B.16}$$

Appendix C: Explicit Casimir Calculations

In this appendix we present the arguments leading to the Poisson brackets (4.21) and (4.22). We then use these results to derive explicit formulae for the Poisson brackets of the Casimirs C^n as defined in (4.17).

First we need the Poisson brackets of the group invariant objects, Q^n, defined in (4.18). We observe that due to the invariance of the trace under cyclic permutations

$$f_{abc}J^a\,\mathrm{Tr}(J^nT^c)=\mathrm{Tr}(J^n[J,T_b])=\mathrm{Tr}(J^{n+1}T_b-J^nT_bJ)=0, \tag{C.1}$$

the Poisson brackets of the Q^n's are given by

$$\{Q^n(x),Q^m(y)\}=\kappa g_{ab}\,\mathrm{Tr}(J^{n-1}T^a)(\mathrm{Tr}(J^{m-1}T^b)(x)\,\delta'+\mathrm{Tr}(J^{m-1}T^b)'\,(x)\,\delta), \tag{C.2}$$

where the argument of δ and δ' is (x^1-y^1). Now by using the identity

$$J^n=\mathrm{Tr}(J^nT_a)T^a+\frac{n}{N}Q^n, \tag{C.3}$$

valid for the A, B, and C series, we find

$$g_{ab}\,\mathrm{Tr}(J^nT^a)\,\mathrm{Tr}(J^mT^b)=(n+m)Q^{n+m}-\frac{nm}{N}Q^nQ^m$$
$$g_{ab}\,\mathrm{Tr}(J^nT^a)\,\mathrm{Tr}(J^mT^b)'=m(Q^{n+m})'-\frac{nm}{N}Q^n(Q^m)'.$$

Together with (C.2) this leads to the Poisson brackets (4.19).

Now we are ready to calculate the Poisson brackets for the generating function for the A_l algebras

$$f(\mu,x)=\log\det(1-\mu J(x)). \tag{C.4}$$

By using the power expansion (4.20) and the Poisson brackets (4.19) one arrives at

$$\{f(\mu, x), f(\nu, y)\} = \kappa \sum_{m,n \geqslant 2} \mu^n \nu^m \left((p-2) Q^{p-2} - \frac{q}{N} Q^{n-1} Q^{m-1} \right)(x)\, \delta' + \kappa \sum_{n,m \geqslant 2} \mu^n \nu^m \left((m-1)(Q^{p-2})' - \frac{q}{N} Q^{n-1} (Q^{m-1})' \right)(x)\, \delta, \tag{C.5}$$

where $p = m + n$ and $q = (n-1)(m-1)$.

The sums quadratic in the Q^n are readily expressed in terms of f and its derivatives and we turn to the more difficult task of expressing the two remaining sums linear in the Q^n in terms of f.

In the first sum

$$\sum_{n,m \geqslant 2} (p-2)\, \mu^n \nu^m Q^{p-2} = \sum_{p \geqslant 4} (p-2)(\mu\nu)^{p/2}\, Q^{p-2} \sum_{m+n=p} (\mu/\nu)^{(n-m)/2}$$

we insert the identity

$$\sum_{n+m=p} \alpha^{n-m} = \frac{\alpha^{p-3} - \alpha^{3-p}}{\alpha - 1/\alpha}$$

and then the remaining sum over p can be written in terms of f and its derivatives as

$$\sum (p-2)\, \mu^n \nu^m Q^{p-2} = \frac{\mu^2 \nu^2}{\mu - \nu} (\partial_\nu f - \partial_\mu f). \tag{C.6}$$

In the second non-trivial sum in (C.5)

$$\sum (m-1)\, \mu^n \nu^m (Q^{p-2})' = \sum (\mu\nu)^{p/2}\, (Q^{p-2})' \sum_{m+n=p} (m-1)(\mu/\nu)^{(n-m)/2}$$

we insert

$$\sum (m-1) \alpha^{n-m} = \frac{\alpha^{p-2} - \alpha^{2-p}}{(\alpha - 1/\alpha)^2} - (p-2) \frac{\alpha^{3-p}}{\alpha - 1/\alpha}$$

and the remaining sum over p can again be written in term of f and its derivatives as

$$\sum (m-1)\, \mu^n \nu^m (Q^{p-2})' = \frac{\mu^2 \nu^2}{\mu - \nu}\, \partial_x\, \partial_\nu f + \frac{\mu^2 \nu^2}{(\mu - \nu)^2}\, \partial_x (f(\nu) - f(\mu)). \tag{C.7}$$

Finally, using (C.6) and (C.7) in (C.5) yields

$$\{f(\mu, x), f(\nu, y)\} = \kappa\mu^2\nu^2 \left(\frac{1}{\mu-\nu}\left[\partial_\nu f - \partial_\mu f\right] - \frac{1}{N}\partial_\mu f\,\partial_\nu f\right)\delta'$$
$$+\kappa\mu^2\nu^2\left(\frac{1}{\mu-\nu}\partial_x\,\partial_\nu f + \frac{1}{(\mu-\nu)^2}\partial_x[f(\nu)-f(\mu)]\right.$$
$$\left.-\frac{1}{N}\partial_\mu f\,\partial_x\,\partial_\nu f\right)\delta. \qquad \text{(C.8)}$$

From this equation one immediately obtains the Poisson brackets (4.21) of the generating polynomial $A(\mu, x) = \exp(f(\mu, x))$ for the A-series.

The Poisson brackets for the generating polynomial of the B and C series can be calculated in a similar manner. The only difference is that instead of formulae (C.6) and (C.7) one needs the identities

$$\sum_{n,m\geqslant 1} 2(p-1)\,\mu^n\nu^m Q^{2(p-1)} = \frac{\mu\nu}{\mu-\nu}(\nu\,\partial_\nu g - \mu\,\partial_\mu g) \qquad \text{(C.9)}$$

and

$$\sum_{n,m\geqslant 1}(2m-1)\,\mu^n\nu^m Q^{2(p-1)} = \frac{1}{2}\mu\nu\frac{\mu+\nu}{(\mu-\nu)^2}(g(\nu)-g(\mu)) + \frac{\mu\nu}{\mu-\nu}\nu\,\partial_\nu g \qquad \text{(C.10)}$$

to derive the Poisson brackets (4.22). (Here $g(\mu, x) = \log B(\mu, x)$.)

The Poisson brackets of the generating polynomials contain all information about the Casimir algebra. For example, by using the expansion (4.17a) in (4.21) one obtains for A_l

$$\{C^k(x), C^k(y)\} = \kappa a_k(x)\,\delta'(x^1-y^1) + \tfrac{1}{2}\kappa a_k'(x)\,\delta(x^1-y^1), \qquad k=1, 2, ..., l, \qquad \text{(C.11)}$$

where

$$a_k = 2k\theta(l+1-2k)C^{2k-1} + \theta(k-2)\cdot k\left(1-\frac{k}{l+1}\right)(C^{k-1})^2$$
$$-2\theta(k-3)\sum_{i=0}^{k-3}\theta(l-i-k)(i+1)\,C^{k+i}C^{k-i-2},$$

and by using the expansion (4.17b) in (4.22) one obtains for the B_l and C_l algebras

$$\{C^k(x), C^k(y)\} = \kappa b_k(x)\,\delta'(x^1-y^1) + \tfrac{1}{2}\kappa b_k'(x)\,\delta(x^1-y^1), \qquad k=1, 2, ..., l, \qquad \text{(C.12)}$$

where

$$b_k = 4(2k-1)\,\theta(l+1-2k)\,C^{2k-1} - 4\theta(k-2) \times \sum_{i=0}^{k-2} \theta(l-i-k)(2i+1)\,C^{k+i}C^{k-i-1}.$$

In particular, for the highest Casimirs the Poisson brackets simplify to (4.23) and (4.24).

ACKNOWLEDGMENT

A. Wipf thanks the Max-Planck-Institut (Werner Heisenberg Institut) where he spent most of the time during which this work was done.

REFERENCES

1. For a review see, e.g., P. GODDARD AND D. OLIVE, *Int. J. Mod. Phys. A* **1** (1986), 303.
2. E. WITTEN, *Commun. Math. Phys.* **92** (1984), 483.
3. P. GODDARD, A. KENT, AND D. OLIVE, *Phys. Lett. B* **152** (1985), 88.
4. A. B. ZAMOLODCHIKOV, *Theor. Math. Phys.* (*Engl. Transl.*) **65** (1986), 1205; V. A. FATEEV AND A. B. ZAMOLODCHIKOV, *Nucl. Phys. B* **280** (1987), 644.
5. V. A. FATEEV AND S. L. LUK'YANOV, *Int. J. Mod. Phys. A* **3** (1988), 507; Kiev preprints ITP-88-74P, ITP-88-75P, ITP-88-76P; S. L. LUK'YANOV, *Funct. Anal. Appl.* **22** (1989), 255.
6. K. YAMAGISHI, *Phys. Lett. B* **205** (1988), 466; P. MATHIEU, *Phys. Lett. B* **208** (1988), 101; I. BAKAS, *Phys. Lett. B* **213** (1988), 313.
7. M. BERSHADSKY AND H. OOGURI, Princeton preprint IASSNS-HEP-89/09; I. BAKAS, *Phys. Lett. B* **219** (1989), 283; *Commun. Math. Phys.* **123** (1989), 627; S. MIZOGUCHI, *Phys. Lett. B* **222** (1989), 226; Y. MATSUO, *Phys. Lett. B* **227** (1989), 209; A. BILAL, *Phys. Lett. B* **227** (1989), 406; CERN preprint TH.5448/89; G. M. T. WATTS, Cambridge preprint DAMPTP 89-08; A. MARSHAKOV AND A. MOROZOV, "A Note on W_3 Algebra," Moscow ITEP preprint (1989); A. MOROZOV, "On the Concept of Universal W-Algebra," Moscow ITEP preprint (1989); Q-HAN PARK, Maryland preprint UMDEPP 90-037 (1989).
8. A. BILAL AND J.-L. GERVAIS, *Phys. Lett. B* **206** (1988), 412; *Nucl. Phys. B* **314** (1989), 646; *Nucl. Phys. B* **318** (1989), 579.
9. O. BABELON, *Phys. Lett. B* **215** (1988), 523.
10. For a review see, e.g., V. DRINFELD AND V. SOKOLOV, *J. Sov. Math.* **30** (1984), 1975.
11. A. M. POLYAKOV, *Mod. Phys. Lett. A* **2** (1987), 893.
12. P. FORGÁCS, A. WIPF, J. BALOG, L. FEHÉR, AND L. O'RAIFEARTAIGH, *Phys. Lett. B* **227** (1989), 214.
13. For quantum Liouville and Toda theories see, e.g., T. L. CURTRIGHT AND C. B. THORN, *Phys. Rev. Lett.* **48** (1982), 1309; E. BRAATEN, T. L. CURTRIGHT, AND C. B. THORN, *Ann. Phys.* (*N.Y.*) **147** (1983), 365; E. D'HOKER AND R. JACKIW, *Phys. Rev. D* **26** (1982); 3517; *Phys. Rev. Lett.* **50** (1983), 1719; J.-L. GERVAIS AND A. NEVEU, *Nucl. Phys. B* **199** (1982), 59; **209** (1982), 125; **224** (1983), 329; **238** (1984), 125; J.-L. GERVAIS, *Nucl. Phys.* (*Proc. Suppl.*) *B* **5** (1988), 119; H. J. OTTO AND G. WEIGHT, *Z. Phys. C* **31** (1986), 219; P. MANSFIELD, *Nucl. Phys. B* **208** (1982), 277; **222** (1983), 419; T. HOLLOWOOD AND P. MANSFIELD, Oxford preprint OUTP-89-20P; T. HOLLOWOOD, Oxford preprint OUTP-89-32P.
14. K. GAWEDZKI AND A. KUPIAINEN, *Phys. Lett. B* **215** (1988), 119; *Nucl. Phys.* **320** (1989), 625; P. BOWCOCK, *Nucl. Phys. B* **316** (1989), 80; D. KARABALI, *Brandeis preprint* BRX TH-275 (1989).

15. F. A. Bais, P. Bouwknegt, M. Surridge, and K. Schoutens, *Nucl. Phys.* **304** (1988), 348, 371; J. Thierry-Mieg, Cambridge preprint DAMPT 87-29.
16. A. N. Leznov and M. V. Savaliev, *Lett. Math. Phys.* **3** (1979), 489; *Commun. Math. Phys.* **74** (1980), 111.
17. A. M. Polyakov and P. B. Wiegmann, *Phys. Lett. B* **131** (1983), 121.
18. V. S. Varadarajan, "Lie Groups, Lie Algebras, and their Representations," Prenctice–Hall, London, 1974.
19. K. Sundermeyer, *in* "Constrained Dynamics," Lecture Notes in Physics, Vol. 169, Springer-Verlag, Berlin, 1982.
20. R. Gilmore, "Lie Groups, Lie Algebras, and Some of their Applications," Wiley, New York, 1974.
21. L. A. Dickey, *Commun. Math. Phys.* **87** (1982), 127.
22. R. Slansky, *Phys. Rep.* **79** (1981), 1.