The Bi–hamiltonian Approach to Integrable Systems

Paolo Casati

Szeged

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Poisson Manifolds

Bihamiltonian manifold are special cases of Poisson manifolds

Definition

A Poisson manifold M is a manifold equipped with a **Poisson bracket**, $\{\cdot,\cdot\}$:

1. *It is a bilinear map:* $\{\cdot\,,\cdot\} : C^\infty(M) \times C^\infty(M) \longrightarrow \mathbb{R}$
2. *Skewsymmetric:* $\{f, g\} = -\{g, f\}$;
3. *It satisfies the Leibniz rule:* $\{f, gh\} = \{f, g\}h + g\{f, h\}$;
4. *It holds the Jacobi identity:* $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$

To a Poisson bracket we can associate at any point of $m \in M$ a Poisson tensor $P_m : T^*_m M \to T_m M$ as follows:

$$\{f, g\}(m) = \langle df_m, P_m dg_m \rangle \qquad \forall f, g \in C^\infty(M)$$

where

- $T_m M$ is the tangent space of M in m;
- $T^*_m M$ is the cotangent space of M in m;
- $\langle \cdot \, , \cdot \rangle$ is the pairing between these two spaces.

Lie algebras as Poisson Manifolds

Let $\mathfrak{g}$ be a (finite dimensional) Lie algebra, and let $\mathfrak{g}^*$ its dual with respect to an ad–invariant bilinear form $\langle \cdot\,, \cdot \rangle$ (which we will be here suppose to be non degenerated).
As vector space $\mathfrak{g}^*$ is a flat manifold and we have obviousely the following identification:

$$T_u\mathfrak{g}^* \simeq T_u^*\mathfrak{g}^* \simeq \mathfrak{g}^* \qquad \forall u \in \mathfrak{g}$$

So for any $f \in C^\infty(\mathfrak{g}^*)$ we can regard its differential df as a linear map between $\mathfrak{g}^*$ and $\mathbb{R}$ i.e., a element of $\mathfrak{g}$. Therefore if $[\cdot\,, \cdot]$ is the Lie bracket of $\mathfrak{g}$ the following bracket on $C^\infty(\mathfrak{g}^*)$ is well defined:

$$\{f, g\}(u) = \langle [df, dg]\,, u \rangle$$

It is easy to show that they are Poisson bracket usually called the Poisson Lie bracket of $\mathfrak{g}$.

Poisson vector Fields

The importance of Poisson manifolds in Physics is that physical dynamical systems are often Poisson vector Fields

Definition

A vector field X on a Poisson manifold (M, P) is said a **Poisson vector field** *if there exist a function $H \in C^\infty(M)$ such that:*

$$X_m = PdH(m) \qquad m \in M.$$

Periodic Toda Lattice for three particles

Let M be the space $\mathbb{R}^3 \times \mathbb{R}^3$ with coordinates (q_i, p_i) and let us consider on it the vector field

$$\begin{aligned} \overset{\circ}{q_i} &= p_i \quad i = 1, 2, 3 \\ \overset{\circ}{p_1} &= e^{q_3 - q_1} - e^{q_1 - q_2} \\ \overset{\circ}{p_2} &= e^{q_1 - q_2} - e^{q_2 - q_3} \\ \overset{\circ}{p_3} &= e^{q_2 - q_3} - e^{q_3 - q_1} \end{aligned}$$

this is a Poisson vector field with Hamiltonian

$$H(p, q) = \sum_{i=1}^{3} p_i^2 + \sum_{i=1}^{3} e^{q_i - q_{i+1}} \qquad i \in \mathbb{Z}_3$$

and the canonical Poisson tensor

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The Korteweg de Vries equation

The Korteweg de Vries equation is the partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3} + 6u\frac{\partial u}{\partial x}$$

where $u \in C^\infty(S^1, \mathbb{R})$. We can regard it as a Poisson vector field on the infinite flat manifold $M = C^\infty(S^1, \mathbb{R})$:

$$\overset{\circ}{u} = P(dH)$$

where

$$P = \partial_x = \frac{\partial}{\partial x} \qquad H = \int_{S^1} (u_{xx} + u^2)dx$$

Definition

A **Bihamiltonian manifold** *M is a manifold equipped with two* **compatible** *Poisson bracket, i.e., two Poisson brackets $\{\cdot,\cdot\}_0$, $\{\cdot,\cdot\}_1$ such that:*
for any $\lambda \in \mathbb{R}$

$$\{\cdot,\cdot\}_\lambda = \{\cdot,\cdot\}_1 - \lambda\{\cdot,\cdot\}_0$$

is a Poisson bracket.

Due to the non linearity of the Jacobi identity this is a non trivial requirement.

Definition

A vector field X on a bihamiltonian manifold M is called a **Bihamiltoinian vector field** *if can be written as*

$$X = P_0(dG) = P_1(dF)$$

for two functions $F, G \in C^\infty(M)$

Bihamiltonian formulation of the Toda Lattice

We introduce Flaschka's variables

$$a_i = \frac{1}{2}e^{(q_i - q_{i+1})/2} \quad b_i = -\frac{1}{2}p_i \quad 1 \le i \le 3, \quad i \in \mathbb{Z}_3 \qquad (1)$$

in order to write the equation of the Toda lattice in the form

$$\begin{aligned} \overset{\circ}{a}_1 &= a_1(b_2 - b_1) & \overset{\circ}{b}_i &= \left((a_1)^2 - (a_3)^2\right) \\ \overset{\circ}{a}_2 &= a_2(b_3 - b_2) & \overset{\circ}{b}_2 &= \left((a_2)^2 - (a_1)^2\right) \\ \overset{\circ}{a}_3 &= a_3(b_1 - b_3) & \overset{\circ}{b}_3 &= \left((a_3)^2 - (a_2)^2\right) \end{aligned} \qquad (2)$$

The equations (2) can be written in the form

$$\overset{\circ}{v} = P_1 w = P_0 z$$

where $v = (a_i, b_i)^T$ and

$$\begin{aligned} w &= (0,0,0,1,1,1)^T = dH_0 \\ H_0 &= b_1 + b_2 + b_3 \\ z &= (a_1, a_2, a_3, b_2 + b_3, b_3 + b_1, b_1 + b_2) = dH_1 \\ H_1 &= \tfrac{1}{2}(a_1^2 + a_2^2 + a_3^2) + b_1 b_2 + b_2 b_3 + b_3 b_1 \end{aligned}$$

the Poissons tensors are

$$P_0 = \begin{pmatrix} 0 & 0 & 0 & a_1 & -a_1 & 0 \\ 0 & 0 & 0 & 0 & a_2 & -a_2 \\ 0 & 0 & 0 & a_3 & 0 & -a_3 \\ -a_1 & 0 & -a_3 & 0 & 0 & 0 \\ a_1 & -a_2 & 0 & 0 & 0 & 0 \\ 0 & a_2 & a_3 & 0 & 0 & 0 \end{pmatrix}$$

$$P_1 = \begin{pmatrix} 0 & -a_1 a_2 & a_1 a_3 & a_1 b_1 & -a_1 b_2 & 0 \\ a_1 a_2 & 0 & -a_2 a_3 & 0 & a_2 b_2 & -a_2 b_3 \\ -a_1 a_3 & a_2 a_3 & 0 & -a_3 b_1 & 0 & a_3 b_3 \\ -a_1 b_1 & 0 & a_3 b_1 & 0 & a_1^2 & -a_3^2 \\ a_1 b_2 & -a_2 b_2 & 0 & -a_1^2 & 0 & a_2^2 \\ 0 & a_2 b_3 & -a_3 b_3 & a_3^2 & -a_2^2 & 0 \end{pmatrix}$$

KdV as Bihamitonian vector field

Let P_0 and P_1 the Poisson tensor Fields

$$P_0 = \partial_x \qquad P_1 = \partial_{xxx} + u\partial_x + 2u_x$$

Then the KdV eqution can be written as bihamitonian vector field as

$$\mathring{u} = P_0 dH_3 = P_1 dH_2$$

with

$$H_2 = \int_{S^1} u dx \qquad H_0 = \int_{S^1} (u_{xx} + \frac{3}{2} u^2) dx$$

Complete Integrability

The Arnold Lioville Theorem roughly states that a Hamiltonian system on a n dimensional manifold M is integrable (can be integrated up to quadrature) if has n Poisson–commuting smooth integrals $F_1, ..., F_n$:

$$\left\{F_i, F_j\right\} = 0, \quad i, j = 1, ..., n,$$

whose differentials are independent in an open dense subset of M.

Bihamitonian manifolds seem to be deeply related to integrable system:

Most of the integrable systems are bihamiltonian.

Why?

Theorem

Suppose that there exists a Casimir of $\{\cdot\,,\cdot\}_\lambda$ *which is a polynomial* $H(\lambda) = \sum_{k=0}^{m} H_k \lambda^k$ *if the system is finite dimensional, or a Laurent series* $H(\lambda) = \sum_{k\leq 1} H_k \lambda^{-k}$ *if the system is infinite dimensional:*

$$\{H(\lambda), F\}_\lambda = 0 \qquad \forall F \in C^\infty(M) \tag{3}$$

then the functions H_k *are conserved quantities in involution:*

$$\left\{H_k, H_j\right\}_\lambda = 0$$

Proof

From the the equation (3) follows immediately

$$\left\{H_j, H_k\right\}_1 = \left\{H_j, H_{k-1}\right\}_0 = -\left\{H_{k-1}, H_j\right\}_0 = -\left\{H_{k-1}, H_{j+1}\right\}_1$$

and consequently if $k > j$

$$\left\{H_j, H_k\right\}_1 = \left\{H_{j+1}, H_{k-1}\right\}_1 = \left\{H_{j+2}, H_{k-2}\right\}_1 = \cdots = \left\{H_k, H_j\right\}_1$$

implying

$$\left\{H_j, H_k\right\}_1 = 0.$$

The same proof show also

$$\left\{H_j, H_k\right\}_0 = 0.$$

Usually one

- find a 1–form $V(\lambda)$ which annihilates the Poisson pencil $P_\lambda = P_1 - \lambda P_0$:
- show that $V(\lambda)$ is closed;
- find a function $H(\lambda)$ such that $V(\lambda) = dH(\lambda)$.

For the Toda lattice we have that there exist two polynomial Casimirs namely: $K = a_1 a_2 a_3$ and $H(\lambda) = H_0\lambda^2 + H_1\lambda + H_2$ with

$$
\begin{aligned}
H_0 &= b_1 + b_2 + b_3 \\
H_1 &= \tfrac{1}{2}(a_1^2 + a_2^2 + a_3^2) + b_1 b_2 + b + 2b_3 + b_3 b_1 \\
H_2 &= -\tfrac{1}{2}(a_1^2 b_3 + a_2^2 b_1 + a_3^2 b_1 + b_1 b_2 b_3)
\end{aligned}
$$

The Casimir $H(\lambda)$ generates two non trivial commuting vector fields:

$$X_1 = P_1 dH_0 = P_0 dH_1 = \begin{pmatrix} a_1 b_1 - a_1 b_2 \\ \text{cyclic permutation} \\ a_1^2 - a_3^2 \\ \text{cyclic permutation} \end{pmatrix}$$

and

$$X_2 = P_1 dH_1 = P_0 dH_2 = \begin{pmatrix} a_1 b_2 b_3 + a_1 a_2^2 - a_1 b_1 b_3 - a_1 a_3^2 \\ \text{cyclic permutation} \\ a_3^2 b_2 - a_1^2 b_3 \\ \text{cyclic permutation} \end{pmatrix}$$

What may be done in the case of an infinite dimensional Bihamiltonian manifold?

- If one of the Poisson tensor is invertible say P_0 then $N^* = P_0^{-1} P_1$ is an adjoint Nijenhuis tensor and

$$V_k = (N^*)^k V_0 \qquad P_1 V_0 = 0$$

is the wanted Casimir N^* adjoint of N.

Otherwise there is no general solution.

Lie algebras as Bihamiltonian Manifolds

Take a Lie algebra $\mathfrak{g}$ and consider its Lie–Poisson tensor P. The tensor P will be depend on the points of $\mathfrak{g}^*$ evaluate it in a point A the pair $P_1 = P$ and $P_0 = P_A$ (P evaluated in A) are a pair of compatible Poisson tensors.

two bihamitonian manifold for thye KdV

The KdV arises as bihamitonian hierachy on the Virasoro algebra with Poisson tensor

$$P_1 = \partial_{xxx} + u\partial_x + 2u_x, \quad P_0 = \partial_x$$

or by reduction of the bihamiltonian hierarchy on the affine Lie algebra $\widehat{\mathfrak{sl}(2)}$ with Poisson tensors

$$(P_1)_S = -\partial_x + [S, \cdot\,] \qquad P_1 = [A, \cdot\,]$$

with

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

The Bihamiltonian reduction Theorem

Theorem

Let $(\mathcal{M}, \{\cdot,\cdot\}_0, \{\cdot,\cdot\}_1)$ *be a bihamiltonian manifold,* $\mathcal{S}$ *a symplectic submanifold of* P_0*, D and E the distributions* $D = P_1 Ker(P_0)$*,* $E = T\mathcal{S} \cap D$*. Then the distribution E is integrable and, if the quotient space* $\mathcal{N} = \mathcal{S}/E$ *is a manifold, it is a bihamiltonian manifold endowed with the reduced Poisson pencil* $\{\cdot,\cdot\}_\lambda^{\mathcal{N}}$ *defined uniquely by the relation*

$$\{f,g\}_\lambda^{\mathcal{N}} \circ \pi = \{F,G\}_\lambda^{\mathcal{M}} \circ i \qquad \forall f,g \in C^\infty(N,\mathbb{C})$$

where i and π *are the canonical injection of* $\mathcal{S}$ *in* $\mathcal{M}$ *and the canonical projection of* $\mathcal{M}$ *onto* $\mathcal{N}$ *respectively, and F and G are any pair of smooth functions, which extend the functions f and g of* $\mathcal{N}$ *into* $\mathcal{M}$*, and are constant on D (i.e.,* $F \circ i = f \circ \pi$ *and* $\{F,K\}_1 = 0$ *for any Casimir K of* P_0*).*

The gauge group for KdV

Theorem

The subspace $\mathfrak{g}_{AB} := \{V \in \mathfrak{g}_A | V_x + [V, B] \in \mathfrak{g}_A^\perp\}$ *(for the KdV* $B = A^T$ *is a subalgebra of* $\mathfrak{g}$ *contained in the nilpotent subalgebra of loops with values in the maximal nilpotent subalgebra* $\mathfrak{n}^- = F(x)A$*. Therefore the corresponding group* $G_{AB} = \exp(\mathfrak{g}_{AB})$ *is well defined. The distribution* E *is spanned by the vector fields* $(P_1)_B(V)$ *with* V *belonging to* $\mathfrak{g}_{AB}$*, and its integral leaves are the orbits of the gauge action of* G_{AB} *on* $\mathcal{S}$ *defined by:*

$$S' = JSJ^{-1} + J_x J^{-1}.$$

The Casimir of the KdV hierarchy

A one form v is Casimir for the Poisson if it satisfies:

$$-\frac{1}{2}v_{xxx} + 2(u = \lambda) + u_x v = 0.$$

If one sets

$$h(x) = \frac{z}{v} + \frac{1}{2}\frac{v_x}{v} \qquad z^2 = \lambda$$

then it can be shown that the Casimir equation it satisfied if and only if $h(x, z)$ satisfies the Riccati equation

$$h_x + h^2 = u + z^2$$

If we write $h(x,z)$ as Laurent series

$$h(x,z) = z + \sum_{k\geq 0} h_k z^{-k}$$

we obtain the recursion relations

$$2h_1 = u$$
$$2h_n + h_{n-1,x} + \sum_{i=1}^{\left[\frac{n-1}{2}\right]} h_i h_{n-1-i} = 0$$

where $[y]$ is the integer part of y. So $h(x)$ can be computed recursively and for KdV equation the Casimir problem can be solved.

That is true for any Drinfeld Sokolov hierarchy but in order to write down the corresponding solvable Riccati equation one must go through the reduction theorem defined above.

separation of variables for the Toda lattice

The Poisson tensor P_0 for the Toda lattice has two Casimirs (K, H_0).
Using the Bihamitonian reduction Theorem we can reduce the system on a symplectic leaf S of P_0.
This reduced system is defined on a Bihamitonian manifold with an invertible Poisson tensor which also called a Nijenhuis Poisson manifold M $(N^* = P_0^{-1} P_1, P_1)$.
In this setting it is possible to "separate the variable".

Let N^* be the adjoint Nijenhuis tensor and $F(x, \lambda)$ a smooth function on S which depends holomorphically from the parameter λ if

$$N^* dF(x, \lambda_i) = \lambda_i dF(x, \lambda_i) \qquad i = 1, \ldots, \dim(S)$$

and

$$Y(F(x, \lambda)) = 1$$

where Y is the vector field associated to the function $\frac{1}{2}\mathrm{Tr}(X)$ by the canonical symplectic form ω then the functions λ_i and the functions $\mu_i = F(x, \lambda_i)$ provide a set of separated coordinates.

In the case of the three periodic Toda lattice those coordinate can be explicitely computed as follows

- Find two vector Z_1 Z_2 fields tranversal to the symplectic foliation of M defined by P_0

$$Z_1 = \partial_{b_3} \qquad Z_2 = \frac{1}{a_1 a_2}\partial_{a_3}$$

- Compute the vector field Y here $Y = a_2\partial_3 - a_3\partial_2$
- The roots of the polynomial

$$Lie_{Z_1}(H(\lambda) = \lambda^2 + (b_1 + b_2)\lambda - \frac{1}{2}(a_1^2 + b_1 b_2)$$

  are the coordinate λ_1, λ_2
- the μ_1, μ_2 are given by the function

$$F = \log\Big(Y(H) + Y^2((H)\Big) = \log\Big((2a_2^2 - a_3^2)\lambda - a_2^2 b_1 + a_3^2 b_2\Big)$$

  evaluated res. in λ_1 and λ_2