

# Szabadsági fokok az Einstein-féle gravitációelméletben

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- 1 The degrees of freedom in GR
- 2 Nested projections and their use
- 3 Solving the constraints
- 4 Summary

# Outline

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# Janus-faced GR: No clear distinction in general relativity

## The arena:

All the pre-GR physical theories provide a distinction between the **arena** in which physical phenomena take place and the **phenomena** themselves.

	arena:	phenomena:
classical mechanics	phase space	dynamical trajectories
electrodynamics	Minkowski spacetime	dynamics of the Maxwell field
general relativity	curved spacetime	the geometry of the spacetime

Such a clear distinction between the arena and the phenomenon is simply not available in general relativity

**the metric plays both roles.**

- GR is more than merely a field theoretic description of gravity. It is a certain body of universal rules:
  - modeling the space of events by a four-dimensional differentiable manifold
  - the use of tensor fields and tensor equations to describe physical phenomena
  - use of the (otherwise dynamical) metric in measuring of distances, areas, volumes, angles ...

# The degrees of freedom in GR:

## What are the degrees of freedom?

- in a theory possessing an initial value formulation: “degrees of freedom” is a synonym of “how many” distinct solutions of the equations exist
- in ordinary particle mechanics: the number of degrees of freedom is the number of quantities that must be specified as initial data divided by two

## The degrees of freedom in the linearized theory:

**Einstein (1916, 1918):** the field equations involve **two degrees of freedom per spacetime point** when studying linearized theory

## Is the full nonlinear theory characterized by two degrees of freedom?

**Darmois (1927):** probably the earliest **answer in the confirmatory** based on consideration of the Cauchy (or initial value) problem

# The degrees of freedom in GR:

## What are the main issues?

- "... **no way singles** out precisely which functions (i.e., which of the 12 metric or extrinsic curvature components or functions of them) can be freely specified, which functions are determined by the constraints, and which functions correspond to gauge transformations. Indeed, **one of the major obstacles to developing a quantum theory of gravity** is the inability to single out the physical degrees of freedom of the theory. "

R.M. Wald: *General Relativity*, Univ. Chicago Press, (1984)

- How these two degrees of freedom may be expressed in terms of the components of the metric tensor and its derivatives (or such combinations of these as, e.g. the Riemann tensor)?
  - Notably, there may be **many possible representations**.
- The main issue is **not to find the only legitimate quantities** representing the gravitational degrees of freedom.
- Rather, **finding a particularly convenient embodiment** of this information which could be used in solving various problems of physical interest.

# The outline:

## Based on three recent papers

- 1 I. Rácz: *Is the Bianchi identity always hyperbolic?*, Class. Quantum Grav. 31 (2014) 155004
- 2 I. Rácz: *Cauchy problem as a two-surface based 'geometrodynamics'*, Class. Quantum Grav. 32 (2015) 015006
- 3 I. Rácz: *Dynamical determination of the gravitational degrees of freedom*, submitted to Class. Quantum Grav.

## The main message:

- 1 smooth Einsteinian spaces of **Euclidean and Lorentzian signature**
  - smoothly foliated by a **two-parameter family of codimension-two-surfaces** which are orientable compact without boundary in  $M$
- 2 the **Bianchi identity** and a **pair of nested  $1 + n$  and  $1 + [n - 1]$  decompositions**
  - **explore the relations** of the various projections of the field equations
- 3 solving the constraints:
  - since the early works by **A. Lichnerowicz (1944)** and **J.W. York (1972)** the constraints are solved by transforming them into a semilinear elliptic system by applying the "conformal method" ... as opposed to this ...
  - the use of some **geometrically distinguished variables** **regardless** whether the primary space is Riemannian or Lorentzian
  - the  $1 + n$  **momentum constraint** as a **first order symmetric hyperbolic syst..**
  - the **Hamiltonian constraint** as a **parabolic** or an **algebraic** equation
- 4 the true degrees of freedom to gravity?

# Assumptions:

- **The primary space:**  $(M, g_{ab})$ 
  - $M$  :  $n + 1$ -dim. ( $n \geq 3$ ), smooth, paracompact, connected, orientable manifold
  - $g_{ab}$ : smooth Lorentzian or Riemannian metric
- **Einsteinian space:** Einstein's equation restricting the geometry

$$G_{ab} - \mathcal{G}_{ab} = 0$$

with source term  $\mathcal{G}_{ab}$  having a vanishing divergence,  $\nabla^a \mathcal{G}_{ab} = 0$ .

- or, more conventionally,

$$[R_{ab} - \frac{1}{2} g_{ab} R] + \Lambda g_{ab} = 8\pi T_{ab}$$

matter fields satisfying their field equations with energy-momentum tensor  $T_{ab}$  and with cosmological constant  $\Lambda$

$$\mathcal{G}_{ab} = 8\pi T_{ab} - \Lambda g_{ab}$$



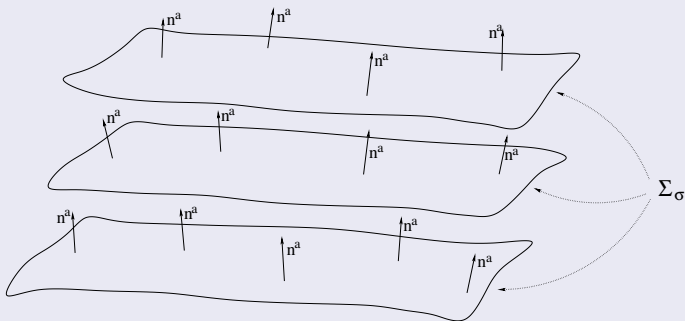
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# The primary $1 + n$ splitting:

No restriction on the topology by Einstein's equations! (local PDEs)

- **Assume:**  $M$  is foliated by a one-parameter family of homologous hypersurfaces, i.e.  $M \simeq \mathbb{R} \times \Sigma$ , for some codimension one manifold  $\Sigma$ .
  - known to **hold for globally hyperbolic spacetimes** (Lorentzian case)
  - **equivalent** to the existence of a smooth function  $\sigma : M \rightarrow \mathbb{R}$  with non-vanishing gradient  $\nabla_a \sigma$  such that the  $\sigma = \text{const}$  level surfaces  $\Sigma_\sigma = \{\sigma\} \times \Sigma$  comprise the one-parameter foliation of  $M$ .



# Projections:

## The projection operator:

- $n^a$  the 'unit norm' vector field that is normal to the  $\Sigma_\sigma$  level surfaces

$$n^a n_a = \epsilon$$

- the sign of the norm of  $n^a$  is not fixed.  $\epsilon$  takes the value  $-1$  or  $+1$  for Lorentzian or Riemannian metric  $g_{ab}$ , respectively.
- **the projection operator**

$$h^a_b = g^a_b - \epsilon n^a n_b \quad [g^a_b \text{ is the identity operator}]$$

to the level surfaces of  $\sigma : M \rightarrow \mathbb{R}$ .

- **the induced metric** on the  $\sigma = \text{const}$  level surfaces

$$h_{ab} = h^e_a h^f_b g_{ef}$$

while  $D_a$  will denote the covariant derivative operator associated with  $h_{ab}$ .

# Decompositions of various fields:

## Examples:

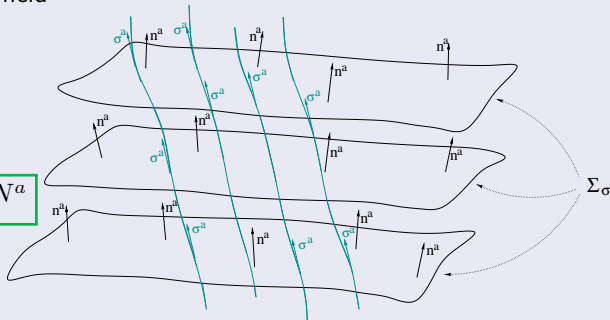
- a form field: 
$$L_a = \delta^e_a L_e = (h^e_a + \epsilon n^e n_a) L_e = \lambda n_a + \mathbf{L}_a$$

- where  $\lambda = \epsilon n^e L_e$  and  $\mathbf{L}_a = h^e_a L_e$

- “time evolution vector field”

$$\sigma^a : \sigma^e \nabla_e \sigma = 1$$

$$\sigma^a = \sigma^a_{\perp} + \sigma^a_{\parallel} = N n^a + N^a$$



- where  $N$  and  $N^a$  denotes the ‘laps’ and ‘shift’ of  $\sigma^a = (\partial_\sigma)^a$ :

$$N = \epsilon (\sigma^e n_e) \quad \text{and} \quad N^a = h^a_e \sigma^e$$

# Decompositions of various fields:

Any symmetric tensor field  $P_{ab}$  can be decomposed

in terms of  $n^a$  and fields living on the  $\sigma = \text{const}$  level surfaces as

$$P_{ab} = \pi n_a n_b + [n_a \mathbf{p}_b + n_b \mathbf{p}_a] + \mathbf{P}_{ab}$$

where  $\pi = n^e n^f P_{ef}$ ,  $\mathbf{p}_a = \epsilon h^e_a n^f P_{ef}$ ,  $\mathbf{P}_{ab} = h^e_a h^f_b P_{ef}$

It is also rewarding to inspect the decomposition of the contraction  $\nabla^a P_{ab}$ :

$$\begin{aligned} \epsilon (\nabla^a P_{ae}) n^e &= \mathcal{L}_n \pi + D^e \mathbf{p}_e + [\pi (K^e_e) - \epsilon \mathbf{P}_{ef} K^{ef} - 2\epsilon \dot{n}^e \mathbf{p}_e] \\ (\nabla^a P_{ae}) h^e_b &= \mathcal{L}_n \mathbf{p}_b + D^e \mathbf{P}_{eb} + [(K^e_e) \mathbf{p}_b + \dot{n}_b \pi - \epsilon \dot{n}^e \mathbf{P}_{eb}] \end{aligned}$$

$$\dot{n}_a := n^e \nabla_e n_a = -\epsilon D_a \ln N$$

## Decompositions of various fields:

## Examples:

- the metric

$$g_{ab} = \epsilon n_a n_b + h_{ab}$$

- the “source term”

$$\mathcal{G}_{ab} = n_a n_b \epsilon + [n_a \mathbf{p}_b + n_b \mathbf{p}_a] + \mathfrak{S}_{ab}$$

where  $\epsilon = n^e n^f \mathcal{G}_{ef}$ ,  $\mathbf{p}_a = \epsilon h^e{}_a n^f \mathcal{G}_{ef}$ ,  $\mathfrak{S}_{ab} = h^e{}_a h^f{}_b \mathcal{G}_{ef}$

- the r.h.s. of our basic field equation  $E_{ab} = G_{ab} - \mathcal{G}_{ab}$

$$E_{ab} = n_a n_b E^{(\mathcal{H})} + [n_a E_b^{(\mathcal{M})} + n_b E_a^{(\mathcal{M})}] + (E_{ab}^{(\mathcal{E}\nu\mathcal{O}\mathcal{L})} + h_{ab} E^{(\mathcal{H})})$$

$$E^{(\mathcal{H})} = n^e n^f E_{ef}, \quad E_a^{(\mathcal{M})} = \epsilon h^e{}_a n^f E_{ef}, \quad E_{ab}^{(\mathcal{E}\nu\mathcal{O}\mathcal{L})} = h^e{}_a h^f{}_b E_{ef} - h_{ab} E^{(\mathcal{H})}$$

# Relations between various parts of the basic equations:

The decomposition of the covariant divergence  $\nabla^a E_{ab} = 0$  of  $E_{ab} = G_{ab} - \mathcal{G}_{ab}$ :

$$\begin{aligned} \mathcal{L}_n E^{(\mathcal{H})} + D^e E_e^{(\mathcal{M})} + [E^{(\mathcal{H})} (K^e_e) - 2\epsilon (\dot{n}^e E_e^{(\mathcal{M})}) \\ - \epsilon K^{ae} (E_{ae}^{(\mathcal{E}\mathcal{V}\mathcal{O}\mathcal{L})} + h_{ae} E^{(\mathcal{H})})] = 0 \\ \mathcal{L}_n E_b^{(\mathcal{M})} + D^a (E_{ab}^{(\mathcal{E}\mathcal{V}\mathcal{O}\mathcal{L})} + h_{ab} E^{(\mathcal{H})}) + [(K^e_e) E_b^{(\mathcal{M})} + E^{(\mathcal{H})} \dot{n}_b \\ - \epsilon (E_{ab}^{(\mathcal{E}\mathcal{V}\mathcal{O}\mathcal{L})} + h_{ab} E^{(\mathcal{H})}) \dot{n}^a] = 0 \end{aligned}$$

a first order symmetric hyperbolic linear homogeneous system for  $(E^{(\mathcal{H})}, E_i^{(\mathcal{M})})^T$

## Theorem

Let  $(M, g_{ab})$  be as specified above. Assume that the metric  $h_{ab}$  induced on the  $\sigma = \text{const}$  level surfaces is Riemannian. Then, **regardless whether  $g_{ab}$  is of Lorentzian or Euclidean signature**, any solution to the reduced equations  $E_{ab}^{(\mathcal{E}\mathcal{V}\mathcal{O}\mathcal{L})} = 0$  is also a solution to the full set of field equations  $G_{ab} - \mathcal{G}_{ab} = 0$  provided that the constraint expressions  $E^{(\mathcal{H})}$  and  $E_a^{(\mathcal{M})}$  vanish on one of the  $\sigma = \text{const}$  level surfaces.

# Relations between various parts of the basic equations:

## Corollary

If the constraint expressions  $E^{(\mathcal{H})}$  and  $E_a^{(\mathcal{M})}$  vanish on all the  $\sigma = \text{const}$  level surfaces then the relations

$$\begin{aligned} K^{ab} E_{ab}^{(\mathcal{E}\mathcal{V}\mathcal{O}\mathcal{L})} &= 0 \\ D^a E_{ab}^{(\mathcal{E}\mathcal{V}\mathcal{O}\mathcal{L})} - \epsilon \dot{n}^a E_{ab}^{(\mathcal{E}\mathcal{V}\mathcal{O}\mathcal{L})} &= 0 \end{aligned}$$

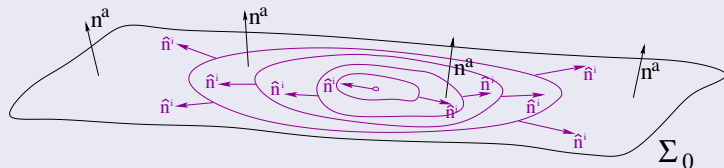
hold for the evolutionary expression  $E_{ab}^{(\mathcal{E}\mathcal{V}\mathcal{O}\mathcal{L})}$ .



# The secondary $1 + [n - 1]$ splitting:

Assume now that on one of the  $\sigma = \text{const}$  level surfaces—say on  $\Sigma_0$ —there exists a smooth function  $\rho : \Sigma_0 \rightarrow \mathbb{R}$ , with nowhere vanishing gradient such that:

- the  $\rho = \text{const}$  level surfaces  $\mathcal{S}_\rho$  are homologous to each other and such that they are orientable compact without boundary in  $M$ .



- The metric  $h_{ij}$  on  $\Sigma_0$  can be decomposed as

$$h_{ij} = \hat{\gamma}_{ij} + \hat{n}_i \hat{n}_j$$

in terms of the positive definite metric  $\hat{\gamma}_{ij}$ , induced on the  $\mathcal{S}_\rho$  hypersurfaces,

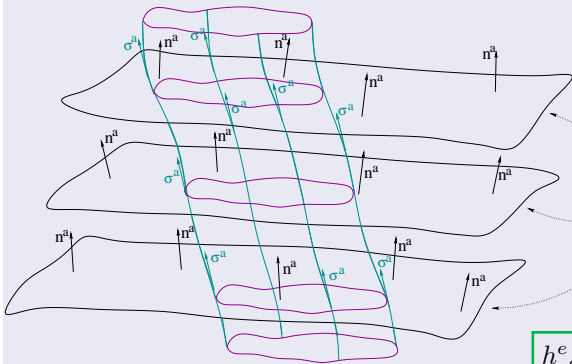
- and the unit norm field

$$\hat{n}^i = \hat{N}^{-1} [(\partial_\rho)^i - \hat{N}^i]$$

normal to the  $\mathcal{S}_\rho$  hypersurfaces on  $\Sigma_0$ , where  $\hat{N}$  and  $\hat{N}^i$  denotes the 'laps' and 'shift' of an 'evolution' vector field  $\rho^i = (\partial_\rho)^i$  on  $\Sigma_0$ .

# Secondary projections:

The Lie transport of this foliation of  $\Sigma_0$  along the integral curves of the vector field  $\sigma^a$  yields then a two-parameter foliation  $\mathcal{S}_{\sigma,\rho}$ :



- the fields  $\hat{n}^i$ ,  $\hat{\gamma}_{ij}$  and the associated projection op.

$\hat{\gamma}^k_l = h^k_l - \hat{n}^k \hat{n}_l$  to the codimension-two surfaces  $\mathcal{S}_{\sigma,\rho}$  get to be well-defined throughout  $M$ .

$\Sigma_\sigma$

- with some algebra

$$h^e_a h^f_b E_{ef} = E_{ab}^{(\mathcal{E}\mathcal{V}\mathcal{O}\mathcal{L})} + h_{ab} E^{(\mathcal{H})}$$

- can be put into the form

$$h^e_a h^f_b E_{ef} = {}^{(n)}E_{ij} = {}^{(n)}G_{ij} - {}^{(n)}\mathcal{G}_{ij}$$

# The integrability condition for ${}^{(n)}G_{ij} - {}^{(n)}\mathcal{G}_{ij} = 0$

$${}^{(n)}E_{ab} = \hat{E}^{(\mathcal{H})} \hat{n}_a \hat{n}_b + [\hat{n}_a \hat{E}_b^{(\mathcal{M})} + \hat{n}_b \hat{E}_a^{(\mathcal{M})}] + (\hat{E}_{ab}^{(\varepsilon\nu\circ\mathcal{L})} + \hat{\gamma}_{ab} \hat{E}^{(\mathcal{H})})$$

$$\hat{E}^{(\mathcal{H})} = \hat{n}^e \hat{n}^f {}^{(n)}E_{ef}, \quad \hat{E}_a^{(\mathcal{M})} = \hat{\gamma}^e{}_a \hat{n}^f {}^{(n)}E_{ef}, \quad \hat{E}_{ab}^{(\varepsilon\nu\circ\mathcal{L})} = \hat{\gamma}^e{}_a \hat{\gamma}^f{}_b {}^{(n)}E_{ef} - \hat{\gamma}_{ab} \hat{E}^{(\mathcal{H})}$$

## Lemma

The integrability condition  $D^a[{}^{(n)}\mathcal{G}_{ab}] = 0$  holds on  $\Sigma_\sigma$  if the momentum constraint expression  $E_b^{(\mathcal{M})}$ , along with its Lie derivative  $\mathcal{L}_n E_b^{(\mathcal{M})}$ , vanishes there.

## Lemma

$\hat{E}^{(\mathcal{H})}$  and  $\hat{E}_a^{(\mathcal{M})}$  vanish identically along a worldline representing a regular origin.

## Corollary

Assume that  $E_b^{(\mathcal{M})} = 0$  on all the  $\Sigma_\sigma$  level surfaces and a regular origin exist in  $M$ . Then any solution to the secondary reduced equations  $\hat{E}_{ij}^{(\varepsilon\nu\circ\mathcal{L})} = 0$  is also a solution to the full set of secondary equations  ${}^{(n)}G_{ij} - {}^{(n)}\mathcal{G}_{ij} = 0$ . ⬅ Theorem

# The explicit forms:

Expressions in the  $1 + n$  decomposition:

$$\begin{aligned}
 E^{(\mathcal{H})} &= n^e n^f E_{ef} = \frac{1}{2} \{ -\epsilon {}^{(n)}R + (K^e_e)^2 - K_{ef} K^{ef} - 2\mathbf{e} \} \\
 E_a^{(\mathcal{M})} &= h^e_a n^f E_{ef} = D_e K^e_a - D_a K^e_e - \epsilon \mathbf{p}_a \\
 E_{ab}^{(\mathcal{E} \vee \mathcal{O} \mathcal{L})} &= {}^{(n)}R_{ab} + \epsilon \{ -\mathcal{L}_n K_{ab} - (K^e_e) K_{ab} + 2 K_{ae} K^e_b - \epsilon N^{-1} D_a D_b N \} \\
 &\quad + \frac{1+\epsilon}{(n-1)} h_{ab} E^{(\mathcal{H})} - \left( \mathfrak{S}_{ab} - \frac{1}{n-1} h_{ab} [\mathfrak{S}_{ef} h^{ef} + \epsilon \mathbf{e}] \right)
 \end{aligned}$$

where

$$\mathbf{e} = n^e n^f \mathcal{G}_{ef}, \quad \mathbf{p}_a = \epsilon h^e_a n^f \mathcal{G}_{ef}, \quad \mathfrak{S}_{ab} = h^e_a h^f_b \mathcal{G}_{ef}$$

and the extrinsic curvature  $K_{ab}$  which is defined as

$$K_{ab} = h^e_a \nabla_e n_b = \frac{1}{2} \mathcal{L}_n h_{ab}$$

where  $\mathcal{L}_n$  stands for the Lie derivative with respect to  $n^a$

# The explicit forms:

Expressions in the  $1 + [n - 1]$  decomposition:

$$\begin{aligned}\hat{E}^{(\mathcal{H})} &= \frac{1}{2} \{-\hat{R} + (\hat{K}^l_l)^2 - \hat{K}_{kl}\hat{K}^{kl} - 2\hat{e}\}, \\ \hat{E}_i^{(\mathcal{M})} &= \hat{D}^l \hat{K}_{li} - \hat{D}_i \hat{K}^l_l - \hat{p}_i, \\ \hat{E}_{ij}^{(\mathcal{E} \vee \mathcal{O} \mathcal{L})} &= \hat{R}_{ij} - \mathcal{L}_{\hat{n}} \hat{K}_{ij} - (\hat{K}^l_l) \hat{K}_{ij} + 2 \hat{K}_{il} \hat{K}^l_j - \hat{N}^{-1} \hat{D}_i \hat{D}_j \hat{N} \\ &\quad + \hat{\gamma}_{ij} \{\mathcal{L}_{\hat{n}} \hat{K}^l_l + \hat{K}_{kl} \hat{K}^{kl} + \hat{N}^{-1} \hat{D}^l \hat{D}_l \hat{N}\} - [\hat{\mathcal{S}}_{ij} - \hat{e} \hat{\gamma}_{ij}]\end{aligned}$$

where  $\hat{D}_i$ ,  $\hat{R}_{ij}$  and  $\hat{R}$  denote the covariant derivative operator, the Ricci tensor and the scalar curvature of  $\hat{\gamma}_{ij}$ , respectively. The 'hatted' source terms  $\hat{e}$ ,  $\hat{p}_i$  and  $\hat{\mathcal{S}}_{ij}$  and the extrinsic curvature  $\hat{K}_{ij}$  are defined as

$$\hat{e} = \hat{n}^k \hat{n}^l {}^{(n)}\mathcal{G}_{kl}, \quad \hat{p}_i = \hat{\gamma}^k_i \hat{n}^l {}^{(n)}\mathcal{G}_{kl} \quad \text{and} \quad \hat{\mathcal{S}}_{ij} = \hat{\gamma}^k_i \hat{\gamma}^l_j {}^{(n)}\mathcal{G}_{kl}$$

and

$$\hat{K}_{ij} = \hat{\gamma}^l_i D_l \hat{n}_j = \frac{1}{2} \mathcal{L}_{\hat{n}} \hat{\gamma}_{ij}$$

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# The complexity can be reduced

by the following splitting of the induced metric  $\hat{\gamma}_{ij}$

There exist a smooth function  $\Omega : M \rightarrow \mathbb{R}$ —which does not vanish except at an origin where the foliation  $\mathcal{S}_\rho$  smoothly reduces to a point on the  $\Sigma_0$  level surfaces—such that the induced metric  $\hat{\gamma}_{ij}$  can be decomposed as

$$\hat{\gamma}_{ij} = \Omega^2 \gamma_{ij}$$

where  $\gamma_{ij}$  is such that

$$\gamma^{ij}(\mathcal{L}_\rho \gamma_{ij}) = 0$$

on each of the  $\mathcal{S}_\rho$  surfaces.

## What does the second relation mean?

In virtue of

$$\gamma^{ij}(\mathcal{L}_\rho \gamma_{ij}) = \mathcal{L}_\rho \ln[\det(\gamma_{ij})]$$

the determinant is independent of  $\rho$  but may depend on the ‘angular’ coordinates.

# Verifications

## The conformal structure:

- Do the desired smooth function  $\Omega : M \rightarrow \mathbb{R}$  and the metric  $\gamma_{ij}$  exist?
- 

$$\hat{\gamma}^{ij}(\mathcal{L}_\rho \hat{\gamma}_{ij}) = \cancel{\gamma^{ij}(\mathcal{L}_\rho \gamma_{ij})} + (n-1) \mathcal{L}_\rho(\ln \Omega^2)$$

- for any smooth distribution of the induced  $[n-1]$ -metric  $\hat{\gamma}_{ij}$  on the  $\mathcal{S}_\rho$  surfaces one may integrate the above relation along the integral curves of  $\rho^a$  on  $\Sigma_0$ , starting at  $\mathcal{S}_0$ .
- one gets  $\Omega^2 = \Omega^2(\rho, x^3, \dots, x^{n+1})$  as

$$\Omega^2 = \Omega_0^2 \cdot \exp \left[ \frac{1}{n-1} \int_0^\rho (\hat{\gamma}^{ij}(\mathcal{L}_\rho \hat{\gamma}_{ij})) d\tilde{\rho} \right]$$

where  $\Omega_0 = \Omega_0(x^3, \dots, x^{n+1})$  denotes the conformal factor at  $\mathcal{S}_0$ .



# The decomposition of the extrinsic curvature:

The  $\Sigma_\sigma$  hypersurfaces in both cases are spacelike:

$$K_{ij} = \kappa \hat{n}_i \hat{n}_j + [\hat{n}_i \mathbf{k}_j + \hat{n}_j \mathbf{k}_i] + \mathbf{K}_{ij}$$

$$\kappa = \hat{n}^k \hat{n}^l K_{kl} = \hat{n}_k (\mathcal{L}_n \hat{n}^k)$$

$$\mathbf{k}_i = \hat{\gamma}^k{}_i \hat{n}^l K_{kl} = \frac{1}{2} \hat{\gamma}^k{}_i (\mathcal{L}_n \hat{n}_k) - \frac{1}{2} \hat{\gamma}_{ki} (\mathcal{L}_n \hat{n}^k)$$

$$\mathbf{K}_{ij} = \hat{\gamma}^k{}_i \hat{\gamma}^l{}_j K_{kl} = \frac{1}{2} \hat{\gamma}^k{}_i \hat{\gamma}^l{}_j (\mathcal{L}_n \hat{\gamma}_{kl})$$

$$\mathbf{K}^l{}_l = \hat{\gamma}^{kl} \mathbf{K}_{kl} = \frac{1}{2} \hat{\gamma}^{ij} (\mathcal{L}_n \hat{\gamma}_{ij}) = \frac{n-1}{2} \mathcal{L}_n \ln \Omega^2$$

(conformal invariant) projection taking the trace free parts on the  $\mathcal{S}_{\sigma,\rho}$  surfaces:

$$\Pi^{kl}{}_{ij} = \hat{\gamma}^k{}_i \hat{\gamma}^l{}_j - \frac{1}{n-1} \hat{\gamma}_{ij} \hat{\gamma}^{kl} = \gamma^k{}_i \gamma^l{}_j - \frac{1}{n-1} \gamma_{ij} \gamma^{kl}$$

$$\mathring{\mathbf{K}}_{ij} = \mathbf{K}_{ij} - \frac{1}{n-1} \gamma_{ij} (\gamma^{ef} \mathbf{K}_{ef}) \quad \text{and} \quad \mathring{K}_{ij} = \hat{K}_{ij} - \frac{1}{n-1} \gamma_{ij} (\gamma^{ef} \hat{K}_{ef})$$

# The $1 + n$ constraints

The momentum constraint:

$$E_a^{(\mathcal{M})} = h^e_a n^f E_{ef} = D_e K^e_a - D_a K^e_e - \epsilon \mathbf{p}_a = 0 \quad \leftarrow \text{div}$$

$$\begin{aligned} (\hat{K}^l_l) \mathbf{k}_i + \hat{D}^l \hat{\mathbf{K}}_{li} + \boldsymbol{\kappa} \dot{\hat{n}}_i + \mathcal{L}_{\hat{n}} \mathbf{k}_i - \dot{\hat{n}}^l \mathbf{K}_{li} - \hat{D}_i \boldsymbol{\kappa} - \frac{n-2}{n-1} \hat{D}_i (\mathbf{K}^l_l) - \epsilon \mathbf{p}_l \hat{\gamma}^l_i &= 0 \\ \boldsymbol{\kappa} (\hat{K}^l_l) + \hat{D}^l \mathbf{k}_l - \mathbf{K}_{kl} \hat{K}^{kl} - 2 \dot{\hat{n}}^l \mathbf{k}_l - \mathcal{L}_{\hat{n}} (\mathbf{K}^l_l) - \epsilon \mathbf{p}_l \hat{n}^l &= 0 \end{aligned}$$

where  $\hat{n}_k = \hat{n}^l D_l \hat{n}_k = -\hat{D}_k (\ln \hat{N})$

After some algebra in coordinates  $(\rho, x^3, \dots, x^{n+1})$  adopted to the foliation  $\mathcal{S}_{\sigma, \rho}$ :

$$\left\{ \left( \begin{array}{cc} \frac{n-1}{(n-2)\hat{N}} \hat{\gamma}^{AB} & 0 \\ 0 & 1 \end{array} \right) \partial_\rho + \left( \begin{array}{cc} -\frac{(n-1)\hat{N}^K}{(n-2)\hat{N}} \hat{\gamma}^{AB} & -\hat{\gamma}^{AK} \\ -\hat{\gamma}^{BK} & -\hat{N}^K \end{array} \right) \partial_K \right\} \begin{pmatrix} \mathbf{k}_B \\ \mathbf{K}^E_E \end{pmatrix} + \begin{pmatrix} \mathcal{B}^A_{(\mathbf{k})} \\ \mathcal{B}_{(\mathbf{K})} \end{pmatrix} = 0$$

Is a first order **symmetric hyperbolic** system for the vector valued variable

$$(\mathbf{k}_B, \mathbf{K}^E_E)^T$$

where the 'radial coordinate'  $\rho$  plays the role of 'time'. ... with characteristic cone (apart from the surfaces  $\mathcal{S}_\rho$  with  $\hat{n}^i \xi_i = 0$ )  $[\hat{\gamma}^{ij} - (n-1) \hat{n}^i \hat{n}^j] \xi_i \xi_j = 0$

# The $1 + n$ constraints

The Hamiltonian constraint:

$$E^{(n)} = n^e n^f E_{ef} = \frac{1}{2} \{-\epsilon^{(n)} R + (K^e_e)^2 - K_{ef} K^{ef} - 2\epsilon\} = 0$$

using 
$${}^{(n)}R = \hat{R} - \left\{ 2 \mathcal{L}_{\hat{n}}(\hat{K}^l_l) + (\hat{K}^l_l)^2 + \hat{K}_{kl} \hat{K}^{kl} + 2 \hat{N}^{-1} \hat{D}^l \hat{D}_l \hat{N} \right\}$$

$$-\epsilon \hat{R} + \epsilon \left\{ 2 \mathcal{L}_{\hat{n}}(\hat{K}^l_l) + (\hat{K}^l_l)^2 + \hat{K}_{kl} \hat{K}^{kl} + 2 \hat{N}^{-1} \hat{D}^l \hat{D}_l \hat{N} \right\} + 2 \kappa (\mathbf{K}^l_l) + (\mathbf{K}^l_l)^2 - 2 \mathbf{k}^l \mathbf{k}_l - \mathbf{K}_{kl} \mathbf{K}^{kl} - 2\epsilon = 0$$

- $\epsilon = \pm 1$  elliptic equation for  $\Omega$ : using 
$$\hat{K}^l_l = \frac{n-1}{2} \mathcal{L}_{\hat{n}} \ln \Omega^2 - \hat{N}^{-1} \mathbb{D}_k \hat{N}^k$$
 and

$$\hat{\gamma}_{ij} = \Omega^2 \gamma_{ij} \implies \hat{R} = \Omega^{-2} \left[ {}^{(\gamma)}R - (n-2) \left\{ \mathbb{D}^l \mathbb{D}_l \ln \Omega^2 + \frac{(n-3)}{4} (\mathbb{D}^l \ln \Omega^2)(\mathbb{D}_l \ln \Omega^2) \right\} \right]$$

- parabolic equation for  $\hat{N}$ : 
$$\hat{K}^l_l = \hat{N}^{-1} \left[ \frac{n-1}{2} \mathcal{L}_{\hat{n}} \ln \Omega^2 - \hat{D}_k \hat{N}^k \right] \& \hat{N}^{-1} \hat{D}^l \hat{D}_l \hat{N}$$

R. Bartnik (1993), R. Weinstein & B. Smith (2004)

- algebraic equation for  $\kappa$  provided that  $\mathbf{K}^l_l$  does not vanish

# Solving the constraints:

- The twelve independent components of the pair  $(h_{ij}, K_{ij})$  may be replaced by

$$(\hat{N}, \hat{N}^i, \Omega, \gamma_{ij}; \kappa, \mathbf{k}_i, \mathbf{K}^l, \mathring{\mathbf{K}}_{ij})$$

- 

$$\kappa = \mathcal{L}_n \ln \hat{N} \quad \text{and} \quad \mathbf{k}_i = (2\hat{N})^{-1} \hat{\gamma}_{il} (\mathcal{L}_n \hat{N}^l)$$

$$\mathbf{K}^l = \frac{n-1}{2} \mathcal{L}_n \ln \Omega^2 \quad \text{and} \quad \mathring{\mathbf{K}}_{ij} = \frac{1}{2} \Omega^2 \gamma^k{}_i \gamma^l{}_j (\mathcal{L}_n \gamma_{kl})$$

$$(\hat{N}, \hat{N}^i, \Omega, \gamma_{ij}; \mathcal{L}_n \hat{N}, \mathcal{L}_n \hat{N}^l, \mathcal{L}_n \Omega, \mathcal{L}_n \gamma_{ij})$$

- The momentum constraint (satisfying a hyperbolic system) can always be solved as an initial value problem with initial data specified at some  $\mathcal{S}_\rho \subset \Sigma_\sigma$  for the variables  $\mathcal{L}_n \hat{N}^l, \mathcal{L}_n \Omega$ .
- The Hamiltonian constraint:
  - $\ell = \pm 1$  elliptic equation for  $\Omega$ : ill-posed together with the hyp.mom.constr.
  - parabolic equation for  $\hat{N}$ :  
 freely specifiable:  $(\hat{N}^i, \Omega, \gamma_{ij}; \mathcal{L}_n \hat{N}, \mathcal{L}_n \gamma_{ij})$
  - algebraic equation for  $\kappa$ :  
 freely specifiable:  $(\hat{N}, \hat{N}^i, \Omega, \gamma_{ij}; \mathcal{L}_n \gamma_{ij})$

# Outline

- 1 The degrees of freedom in GR
- 2 Nested projections and their use
- 3 Solving the constraints
- 4 Summary

# Summary:

- ① smooth Einsteinian spaces of **Euclidean and Lorentzian signature**  $[n + 1\text{-dim. } (n \geq 3)]$ 
  - smoothly foliated by a **two-parameter family of codimension-two-surfaces**:
- ② the **Bianchi identity** and a **pair of nested  $1 + n$  and  $1 + [n - 1]$  decompositions**
  - **explored relations** of the various projections of the field equations
- ③ solving the constraints:
  - introducing some **geometrically distinguished variables** **regardless** whether the primary space is Riemannian or Lorentzian
  - the  $1 + n$  **momentum constraint** as a **first order symmetric hyperbolic system**.
  - the **Hamiltonian constraint** as a **parabolic** or an **algebraic** equation
- ④ the **conformal structure**  $\gamma_{ij}$ , defined on the foliating codimension-two surfaces  $\mathcal{S}_\rho$ , appears to provide a **convenient embodiment** of the degrees of freedom to various metric theories of gravity

**Thanks for your attention**