A foton fázisoperátora mint Haar-integrál a Blaschke-csoporton.

Varró Sándor[1,2]

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2) ELI Attosecond Light Pulse Source, ELI-HU Nonprofit Kft, Szeged

**SZTE Elm. Fiz. Szeminárium,
2016. október 6.**

- Bevezetés. Kvantum szögváltozók, lineáris oszcillátor.
- Reguláris fázisoperátor, koherens fázisállapotok.
- A reguláris fázisoperátor mint Haar-integrál.
- Összefoglalás.

Kvantum szögváltozók, [Quantum angle-variables; basic and recent canonical references.]

Quantization of complex amplitudes. Number, phase.

Quantization, photon number and phase (?). Dirac (1927). Jordan (1927). London (1927). Fock (1932). 'Half- unitarity' of the 'exponential phase operator'. [E^+ partially isometric, but NOT unitary.]

Dirac: $b_r = e^{-i\theta_r/h} N_r^{1/2}, \quad b_r^* = N_r^{1/2} e^{+i\theta_r/h}$ $\qquad b_r b_r^* - b_r^* b_r = 1$

OPERATOR analogon of the polar decomposition of a complex NUMBER: $z = e^{i\varphi}\sqrt{z^* z} = |z| e^{i\varphi}$

$$A = E\sqrt{A^+ A} \neq e^{i\Phi}\sqrt{N}$$

$$EE^+ = 1, \quad E^+ E = 1 - P_0$$

$$E \neq e^{i\Phi}$$

$$EE^+ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \ddots \end{pmatrix}$$

$$E^+ E = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \ddots \end{pmatrix}$$

$$[N, \Phi] \neq i$$

Refinement of the history: Dirac quoted London's work (non-existence of quantum angle). He told Jordan that he used only $E^+=e^{2\pi iw}$, but not w (quantum-angle).

>>As Jordan (1927) noted: „Thus, one is not allowed for instance for the action and angle variables, J and w, respectively, to write Jw-wJ=h/2πi. (5) That it was possible to derive several correct results from this not-allowed equation (5), according to a remark by Dirac, is to be understood so that for the derivation of these results, in fact, instead of (5) only $Je^{2\pi iw}-e^{2\pi iw}J=he^{2\pi iw}$ (6) has been used."<<

$$NE^{+} - E^{+}N = E^{+}$$

>>In the same year, earlier than Dirac's mentioned paper appeared, London (1927) published his study on the angle variables and canonical transformations in quantum mechanics. He proved that though the ladder operators E and E^+ have a well-defined matrix representation, they cannot be expressed as an exponential of the form $e^{i\phi}$, where ϕ would be a hermitian matrix. Dirac (1927) was aware of this discrepancy, he even quoted London's paper (see note after the reference to Dirac's paper).<<

>>Dirac P A M 1927 The quantum theory of emission and absorption of radiation *Proc. Roy. Soc. A* **114**, 243-265. On page 245 Dirac wrote „The mathematical development of the theory has been made possible by the author's general transformation theory of the quantum matrices.‡ In the third footnote on this page: „‡ Roy. Soc. Proc., A, vol.113, p.621 (1927). ... An essentially equivalent theory was developed by Jordan [Z. f. Physik, vol. 40, p. 809 (1927)]. See also, F. London, Z. F. Physik, vol. 40, p. 193 (1926)." <<

[Quotations from our recent paper; **S. V., Regular phase operator and SU(1,1) coherent states of the harmonic oscillator. Physica Scripta 90 (7), 074053 (2015)]**

Selection from earlier references on the quantum action-angle [number-phase] variables.

Dirac P A M 1927 The quantum theory of emission and absorption of radiation . *Proc. Roy. Soc. A* **114**, 243-265.
London F 1926 Über die Jacobischen Transformationen der Quantenmechanik. *Zeitschrift für Physik* **37**, 915-925.
London F 1927 Winkelvariable und kanonische Transformationen in der Undulationsmechanik . *Zeitschrift für Physik* **40**, 193-210.
Jordan P 1927 Über eine neue Begründung der Quantenmechanik. II *Zeitschrift für Physik* **44**, 1-25 See page 3. This paper is a continuation of an earlier paper: Jordan P, *Zeitschrift für Physik* **40**, 809-838 (1927)
Weyl H 1931 *The theory of groups and quantum mechanics* Page 36. (Dover Publications, New York, 1931) (Translated from the second (revised) German edition by Robertson H P)

Susskind L and Glogower J 1964 Quantum mechanical phase and the time operator *Physics* **1**, 49-61.
Carruthers P and Nieto M M 1968 Phase and angle variables in quantum mechanics *Rev. Mod. Phys*. **40**, 411-440
Jackiw R 1968 Minimum uncertainty product, number uncertainty product, and coherent states *J. Math. Phys*. **9**, 339-346
Garrison J C and Wong J 1970 Canonically conjugate pairs, uncertainty relations, and phase operators *J. Math. Phys.* **11**, 2242-2249
Y. Aharonov, E. C. Lerner, H. W. Huang, and J. M. Knight, Oscillator phase states, thermal equilibrium and group representations. *Journal of Mathematical Physics* **14,** 746-756 (1973).
Paul H 1974 Phase of a microscopic electromagnetic field and its measurement *Fortschritte der Physik* **22**, 657.
Popov V N and Yarunin V S 1992 Quantum and quasi-classical states of the photon phase operator. *Journal of Modern Optics* **39**, 1525-1531.

Pegg D T and Barnett S M 1989 Phase properties of the quantized single-mode electromagnetic field *Phys. Rev. A* **39**, 1665-1675.
Schleich W, Horowicz R J and Varró S 1989 A bifurcation in the phase probability distribution of a highly squeezed state *Phys. Rev. A* **40**, 7405-7408.
Schleich W P, Dowling J P, Horowicz R J and Varró *S , in New Frontiers in Quantum Electrodynamics ed* Barut A O (New York: Plenum, 1990)
Shapiro J H and Shepard S R 1991 Quantum phase measurement:.*Phys. Rev. A* **43**, 3795-3818.
Noh J W, Fougères A and Mandel L 1992a Operational approach to the phase of a quantum field *Phys. Rev. A* **45**, 424-442.
Gantsog Ts, Miranowicz A and Tanaś 1992 Phase properties of real field states: The Garrison-Wong versus Pegg-Barnett predictions *Phys. Rev. A* **46** 2870-2876
Schleich W P and Barnett S M 1993 (Editors) Special issue on *Quantum phase and phase dependent measurements Physica Scripta* **T48**
Freyberger M, Heni M and Schleich W P, Two-mode quantum phase. *Quantum Semiclass. Opt.* **7** (1995) 187-203.

Some related basic mathematical references. Explicitly related works.

Neumark M A, Positive definite operator functions on a commutative group. Bulletin (Izvestiya) Acad. Sci. URSS (ser. math) **7**, 237-244 (1943). Self-adjoint extension of the second kind of a symmetric operator. Ibidem, **4** (1940), 53-104. (in Russian with English summary)
Riesz F and Szőkefalvi-Nagy B 1965 *Leçons d'analyse fonctionelle 4. éd.* (Gautier-Villars, Paris and Akadémia Kiadó, Budapest, 1965)
Szőkefalvi-Nagy B, Sur le contractions de l'espace de Hilbert.Acta Sci. Math. Szeged **15**, 87-92 (1953). „Extension of linear transformations in Hilbert space which extend beyond this space." Appendix to F. Riesz and B. Sz.-Nagy: Functional analysis (Ungar, New York, 1960).

Some explicitly related works.

Perelomov A M 1972 *Commun. Math. Phys.* **26** 222***;*** Perelomov A 1986 *Generalized Coherent States and their Applications (Basel: Springer)*
Y. Aharonov, E. C. Lerner, H. W. Huang, and J. M. Knight, Oscillator phase states, thermal equilibrium and group representations. *Journal of Mathematical Physics* **14,** 746-756 (1973).
Holevo A S, *Probabilistic and Statistical Aspects of Quantum Theory* (North-Holland, Amsterdam, 1982)
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Brif C, Photon states associated with the Holstein-Primakoff realization of the SU(1,l) Lie algebra. *Quantum and Semiclassical Optics* **7**, 803-834 (1995).
M Rasetti, A fully consistent Lie algebraic representation of quantum phase and number operators. J. Phys. A: Math. Gen. **37** (2004) L479–L487.
J. Lahti P J and Pellonpa¨a¨ J P, *J. Math. Phys*. **40,** 4688 (1999)
Busch P, Lahti P, J-P Pellonp¨a¨a J-P and K Ylinen, J. Phys. A: Math. Gen. **34** (2001) 5923–5935.
Pellonpa¨a¨ J P, On the structure of covariant phase observables. *J. Math. Phys.* **43,** 1299- (2002)

{ [2] Varró S, Quantum phase operator and projectors based on SU(1,1) coherent states. Talk 7.3.1. presented at Seminar 7 of LPHYS'14: Quantum information and quantum computation [23th International Laser Physics Workshop, 14-18 July 2014, Sofia, Bulgaria }

„Canonical references" in the modern era (from the 60ies of the last century) on the phase operator in quantum optics. (p.1)

Susskind L and Glogower J 1964 Quantum mechanical phase and the time operator *Physics* 1, 49-61.
On page 50:
"Ideally we should like to be able to express a^+ and a^- in the form $Re^{\iota\phi}$ and $e^{\iota\phi}$ R where R is Hermitian and $e^{\iota\phi}$ is a unitary operator defining a Hermitian ϕ . This is what Heitler [2] and Dirac [3] try to do. We shall find, however, that their arguments are not correct."
....
[2] W. HEITLER, *Quantum Theory of Radiation* Chap. II. Oxford University Press (1954).
[3] P. DIRAC, *Quantum Theory of Emission and Absorption in Quantum Electrodynamics* (Edited by J. SCHWINGER) Dover Publications, New York (1958)."

##
Carruthers P and Nieto M M 1968 Phase and angle variables in quantum mechanics *Rev. Mod. Phys*. 40, 411-440. On page 412:
„ Section 5 discusses the extent to which number and phase variables can be used to describe the quantum mechanical harmonic oscillator. In this original paper dealing with the quantization of the electromagnetic field, Dirac[17] assumed the existence of an Hermitian operator. As shown in an important paper by Susskind and Glogower,[18] this assumption leads to contradictions. However, one can describe the phase by means of two well-defined Hermitian operators C and S, which correspond to cosϕ and sinϕ in the classical limit. It is stressed that the the absence of a proper phase operator results from the boundedness of the eigenvalue spectrum of the number operator."

##
Jackiw R 1968 Minimum uncertainty product, number uncertainty product, and coherent states *J. Math. Phys*. 9, 339-346. On page 339 in the Introduction:
"Recent discussions of a quantum-mechanical phase operator for harmonic oscillators have shown that a Hermitian phase operator does not exist.[1] Susskind and Glogower[1] (SG) have demonstrated however that Hermitian sine (S) and cosine (C) operators can be defined which have many properties that are suggested by the nomenclature. Carruters and Nieto[2] (CN) have examined the matrix elements of S and C between Glauber's[3] coherent states. They found that in the high-excitation (classical) limit the expectation values of S and C, in these states, behave as the sine and cosine of the phase of the harmonic oscillaton."

„Canonical references" in the modern era (from the 60ies of the last century) on the phase operator in quantum optics. (p.2)

Garrison J C and Wong J 1970 Canonically conjugate pairs, uncertainty relations, and phase operators *J. Math. Phys.* 11, 2242-2249.
On page 2244, starting section 4. entitled „Phase operator":
„It has often been assumed that the annihilation operator for a harmonic oscillator has the representation $a = e^{-i\Theta} N^{\frac{1}{2}}$ (4.1) in which the number operator N and the phase operator Θ are self-adjoint.

A formal calculation based on this representation shows that (H, N) is a canonically conjugate pair. This simple picture was destroyed by Susskind and Glogower,[1] who proved that no unitary operator exp(-iH) could satisfy (4.1). They replaced this incorrect representation by the rigorous polar decomposition

$a = (N+1)^{\frac{1}{2}} E$ where E is defined by $E\phi_0 = 0$, $E\phi_n = \phi_{n-1}$ (4.2) ,

and $\{\phi_n\}$ is the complete orthonormal set of eigenfunctions of N. Although the relations (4.2) show that E is not unitary, one can introduce useful self-adjoint operators C and S by

$E = C + iS$ (4.3).

These are the cosine and sine operators introduced by Susskind and Glogower.[1]"

##

Y. Aharonov, E. C. Lerner, H. W. Huang, and J. M. Knight 1973, Oscillator phase states, thermal equilibrium and group representations. *Journal of Mathematical Physics* 14, 746-756 (1973).
On page 746 the Introduction starts as:

"In studying the quantum theory of harmonic oscillator phase, new hermitian operators C and S were introduced1-7 whose spectra coincide with the range of valoes of the trigonometric functions cosϕ and sinϕ. Since these operators do not commute with one another, on cannot prepare a state in which the phase is arbitrarily sharply defined except in certain limiting cases. However, one might expect that the operator U=C+iS, which is the quantum analog of the quantity expiϕ=cosϕ+isinϕ, would define states of maximal phase resolution in come reasonable sense."

##

F. London (1927): There is no hermitian phase operator; [N,Φ]=i ‘not good’ !
Getting around: ‘Quantum Cosine and Sine’ (Susskind and Glogower (1964))...

$$A = E\sqrt{N}$$

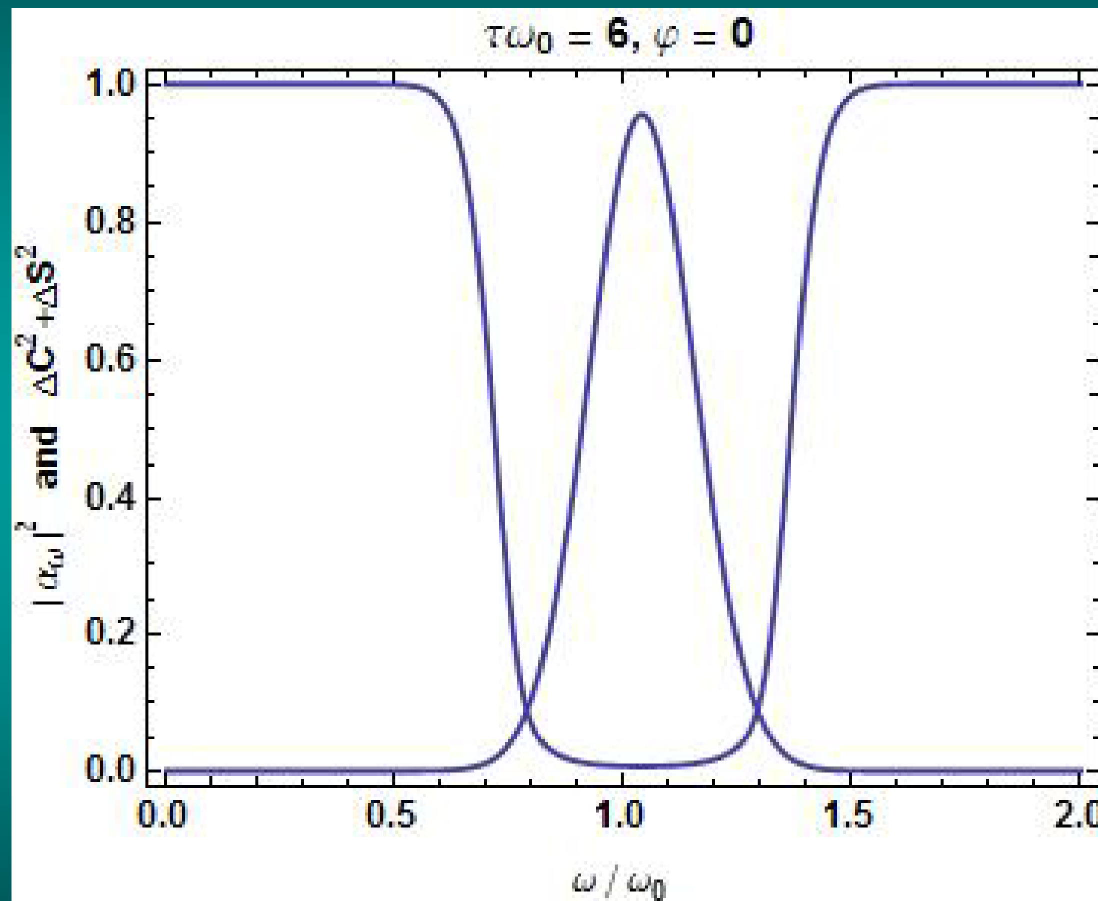
$$C \equiv \frac{1}{2}(E + E^{+})$$

$$S \equiv \frac{1}{2i}(E - E^{+})$$

$$[C, S] \neq 0$$

$$\overline{O} \equiv \langle \alpha_{\omega} | \hat{O} | \alpha_{\omega} \rangle$$

$$\Delta C^{2} \equiv \overline{(\hat{C} - \overline{C})^{2}}$$



S. V., Entangled photon-electron states and the number-phase minimun-uncertainty states of the photon field. New Journal of Physics 10, 053028 (2011); S. V., Entangled states and entropy remnants of a photon-electron system. Physica Scripta T140, 014038 (2010). Varró S; Intensity effects and absolute phase effects in nonlinear laser-matter interactions; In *Laser Pulse Phenomena and Applications* (Ed. Duarte F J); Chapter 12, pp 243-266 . Lecture Notes (in Hung.) Theor. Physics . SZTE (2012).

Garrison and Wong (1970): The solution.

JOURNAL OF MATHEMATICAL PHYSICS VOLUME 11, NUMBER 8 AUGUST 1970

Canonically Conjugate Pairs, Uncertainty Relations, and Phase Operators*

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(Received 1 December 1969)

Apparent difficulties that prevent the definition of canonical conjugates for certain observables, e.g., the number operator, are eliminated by distinguishing between the Heisenberg and Weyl forms of the canonical commutation relations (CCR's). Examples are given for which the uncertainty principle does not follow from the CCR's. An operator F is constructed which is canonically conjugate, in the Heisenberg sense, to the number operator; and F is used to define a quantum time operator.

1. INTRODUCTION

A great deal of effort has been expended in the study of canonical commutation relations and the associated uncertainty principle. In particular, the question of the existence of a phase operator canonically conjugate to the number operator has excited

2. CANONICAL COMMUTATION RELATIONS

The observables Q and P are said to be canonically conjugate if they satisfy the (abstract) canonical commutation relation (CCR)

$$[Q, P] = i. \tag{2.1}$$

$$(f, \Phi_{GW} g) = \int_{\theta_0}^{\theta_0+2\pi} d\theta f^*(e^{i\theta})[\theta \cdot g(e^{i\theta})]$$

Garrison J C and Wong J, Canonically conjugate pairs, uncertainty relations, and phase operators. *Journal of Mathematical Physics* 11 (8), 2242-2249 (1970).

„Canonical references" in the modern era (from the 60ies of the last century) on the phase operator in quantum optics. (p.3)

Pegg D T and Barnett S M 1988, Unitary phase operator in quantum mechanics. *Europhysics Letters* 6(6), 483-487 (1988). The introduction starts as:

"In his original description of the quantized electromagnetic field, Dirac [1] postulated the existence of a Hermitian operator ϕ. This proposed operator would exist in a unitary exponential form exp[if] which, together with the square root of the number operator N, would appear in a decomposition of the creation and annihilation operators. However, difficulties have been found with this postulate. The problem of the multivaluedness was easily overcome [2], but Susskind and Glogower [3] have emphasized the difficulty in actually finding an exponential phase operator which is unitary. Indeed it has been considered for some time that such an operator, with all simple and desirable properties that would make it acceptable as a quantum phase, may not even exist[4]."

##

For example: Cibils M B, Cuche Y, Marvulle V and Wreszinski W F 1991, Connection between the Pegg-Barnett and the Bialynicki-Birula phase operators. *Physical Review A* 43 (7), 4044-4046.

"Unfortunately, since the original description of the quantized electromagnetic field by Dirac where the existence of a Hermitian phase operator was postulated,[2] many difficulties arose with this concept. In particular, Susskind and Glogower[3] showed that, in the usual formulation, the phase operator ϕ is not self-adjoint, or, alternatively, "exp(-iϕ)" is not unitary. This proof may also be found in the excellent review by Carruthers and Nieto:[4]…"

##

And so on....

##

We note that all the basic formulae concerning 'Φ and 'N 'in a finite-dimensional Hilbert space (thus, in the present context: the so-called 'Pegg-Barnett formalism') can be found in the early work of Weyl:

Weyl H 1931 *The theory of groups and quantum mechanics* Page 36. (Dover Publications, New York, 1931) (Translated from the second (revised) German edition by Robertson H P).

Loudon (1973). Pegg and Barnett (1988-89): ‘Phase operator’ in finite dimensional space

PHYSICAL REVIEW A VOLUME 39, NUMBER 4 FEBRUARY 15, 1989

Phase properties of the quantized single-mode electromagnetic field

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(Received 12 September 1988)

The usual mathematical model of the single-mode electromagnetic field is the harmonic oscillator with an infinite-dimensional state space, which unfortunately cannot accommodate the existence of a Hermitian phase operator. Recently we indicated that this difficulty may be circumvented by using an alternative, and physically indistinguishable, mathematical model of the single-mode field involving a finite but arbitrarily large state space, the dimension of which is allowed to tend to infinity *after* physically measurable results, such as expectation values, are calculated. In this paper we investigate the properties of a Hermitian phase operator which follows directly and uniquely from the form of the phase states in this space and find them to be well behaved. The phase-number commutator is not subject to the difficulties inherent in Dirac's original commutator, but still preserves the commutator–Poisson-bracket correspondence for physical field states. In the quantum regime of small field strengths, the phase operator predicts phase properties substantially different from those obtained using the conventional Susskind-Glogower operators. In particular, our results are consistent with the vacuum being a state of random phase and the phases of two vacuum fields being uncorrelated. For higher-intensity fields, the quantum phase properties agree with those previously obtained by phenomenological and semiclassical approaches, where such approximations are valid. We illustrate the properties of the phase with a discussion of partial phase states. The Hermitian phase operator also allows us to construct a unitary number-shift operator and phase-moment generating functions. We conclude that the alternative mathematical description of the single-mode field presented here provides a valid, and potentially useful, quantum-mechanical approach for calculating the phase properties of the electromagnetic field.

$$\theta_m = \theta_0 + \frac{2\pi}{s+1} m$$

$$|\theta\rangle = \frac{1}{\sqrt{s+1}} \sum_{n=0}^{s} e^{in\theta} |n\rangle$$

$$\Phi_{PB} = \sum_{m=0}^{s} |\theta_m\rangle \theta_m \langle\theta_m|$$

Loudon R, *The Quantum Theory of Light*, 1st ed. (Oxford University Press, Oxford, 1973), p. 143. Pegg D T and Barnett S M, Phase properties of the quantized single-mode electromagnetic field. Physical Review A 39 (4) 1665-1675 (1989).

London' paper was first mentioned in the 'modern era' by Schleich, Horowicz and Varró, (1989).

$$P_{\varphi}[|\psi\rangle] = \int_0^{\infty} d\rho\rho W(x = \rho\cos\varphi, y = \rho\sin\varphi; |\psi\rangle)$$

RAPID CO

PHYSICAL REVIEW A VOLUME 40, NUMBER 12 DECEMBE

Rapid Communications

The Rapid Communications section is intended for the accelerated publication of important new results. Since r submitted to this section are given priority treatment both in the editorial office and in production, authors should their submittal letter why the work justifies this special handling. A Rapid Communication should be no longer than pages and must be accompanied by an abstract. Page proofs are sent to authors, but, because of the accelerate publication is not delayed for receipt of corrections unless requested by the author or noted by the editor.

Bifurcation in the phase probability distribution of a highly squeezed state

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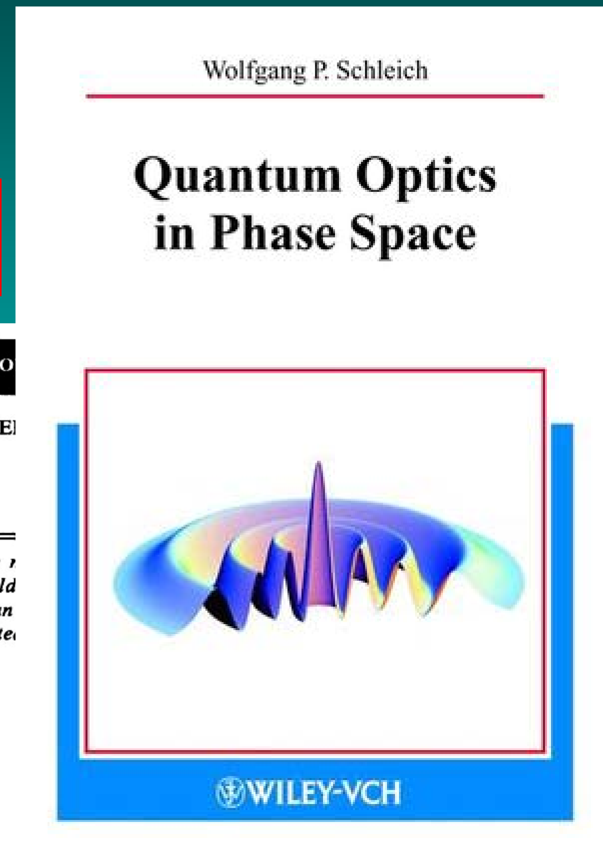
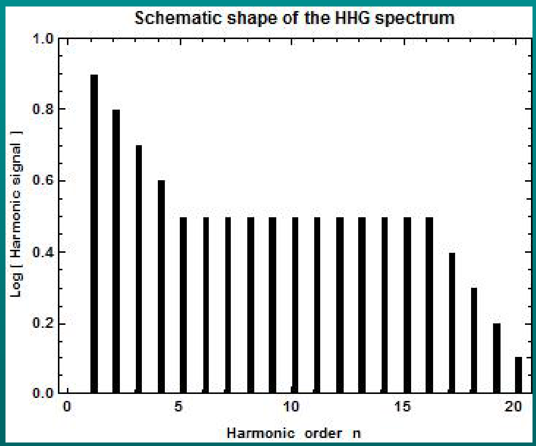
(Received 2 October 1989)

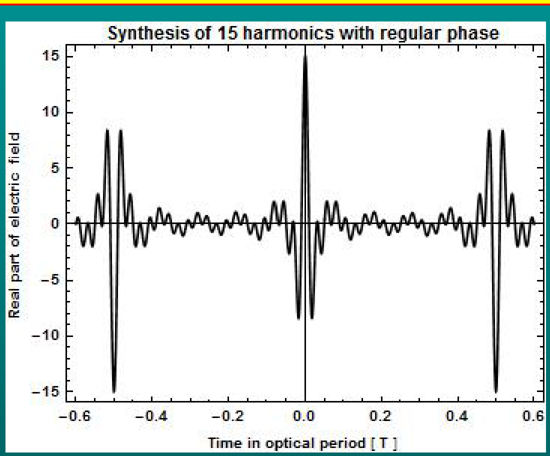
We calculate the phase distribution of a highly squeezed state using both a definition of a phase eigenstate and the area-of-overlap principle. This probability curve undergoes a transition from a single- to a double-peaked distribution when we decrease the product of squeeze and displacement parameters.

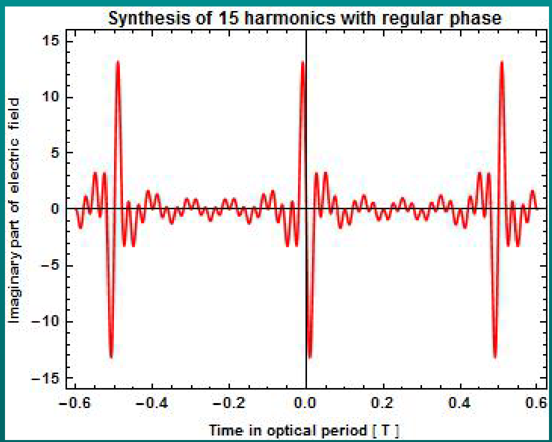


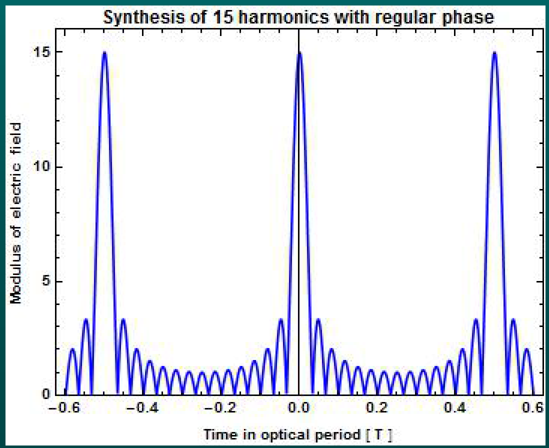
Classical phase-locking of the higher-harmonic components stemming from an interaction of a very high-intensity laser with an atomic jet. [e.g. á la Gy. Farkas & Cs. Tóth (1992)]
Relevance in attosecond physics...?

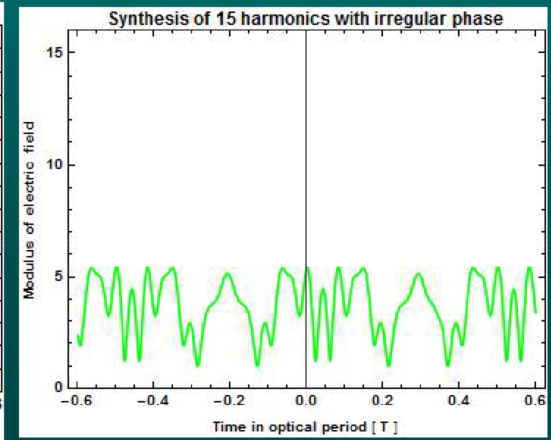
$$E(t) = \sum_{n=n_1=2k_1+1}^{n=n_2=2k_2+1} A_n e^{i\phi_n} e^{-2\pi i n(t/T)} \approx A_{n_1} \sum_{n=n_1}^{n=n_2} e^{-2\pi i n(t/T)+i\phi_n}. \quad (5.12)$$





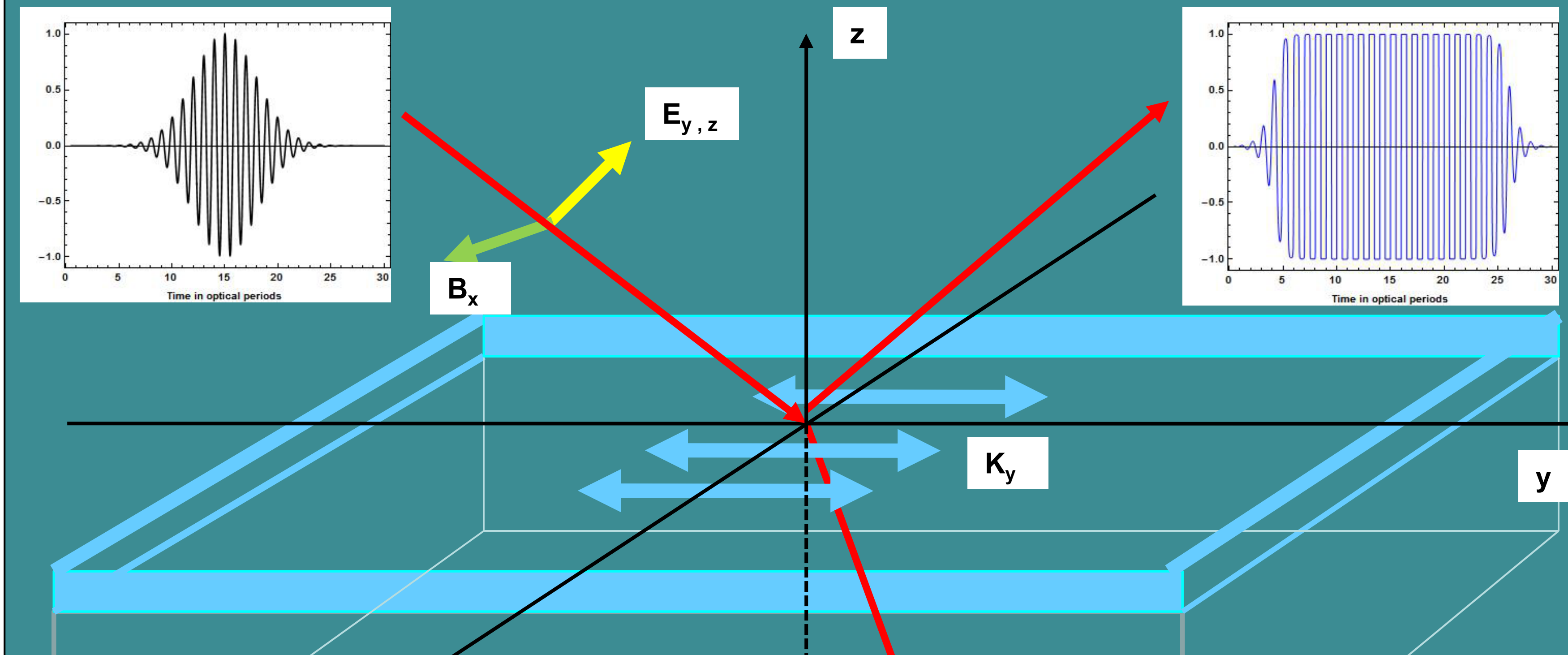


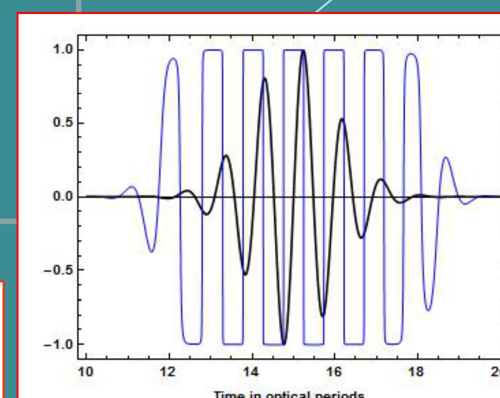




Synthesis of N = 15 higher-harmonics of the same intensity at the plateau region, according to (5.12); Real part, imaginary part and modulus with phase-locking. In the last figure the phase difference of the components are not constant (random).

Scattering of a p-polarized wave on graphene. Generation of an optical Rademacher system.





S. V., Graphene-based carrier-envelope phase difference meter. In *AIP* Conf. *Proceedings 1462, 128-131 (2012)*. Varró S, Generation of rectangular optical waves by relativistic clipping. *Laser Physics* 23 (2013) 056006 (6pp) .

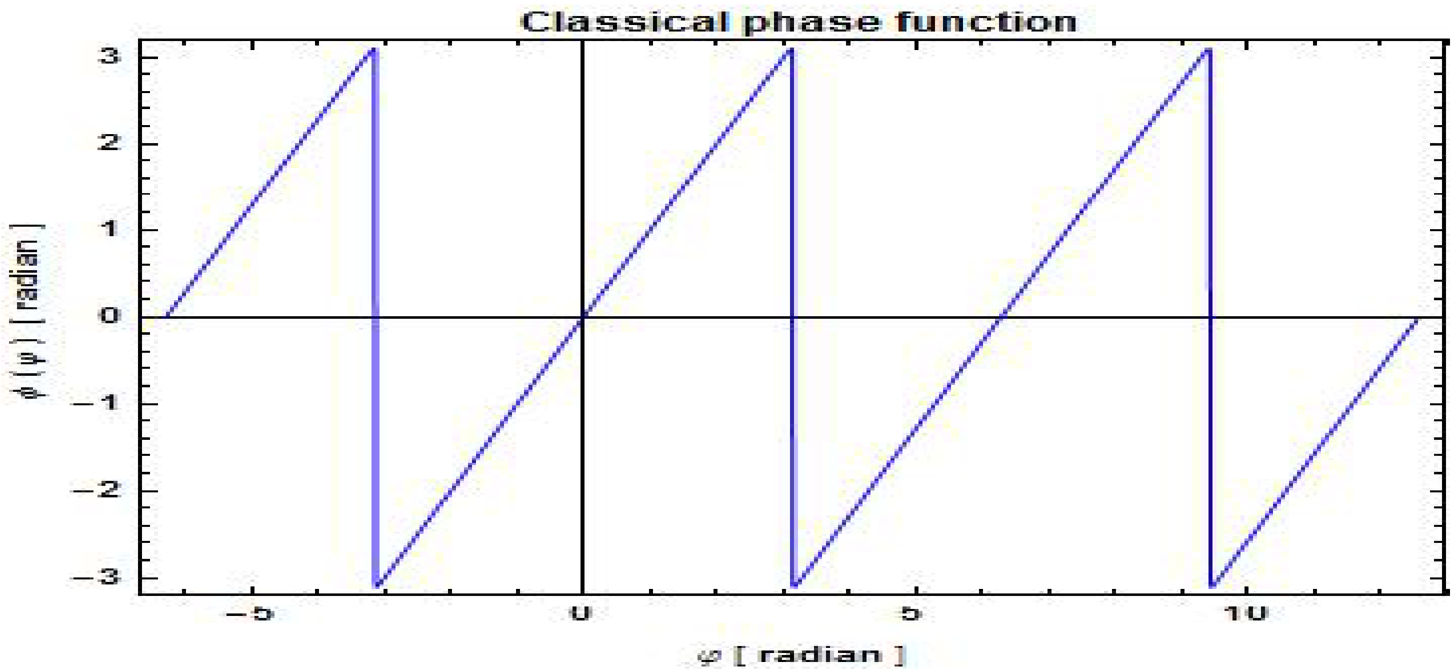
Fourier-korrespondencia, reguláris fázisoperátor, koherens fázisállapotok.

Classical (2π-periodic) phase fuction. 'Saw-tooth': $\Sigma_k$sinkx/k.

$$\Phi_{cl}(e^{i\varphi}) = \varphi_0 + \varphi =$$

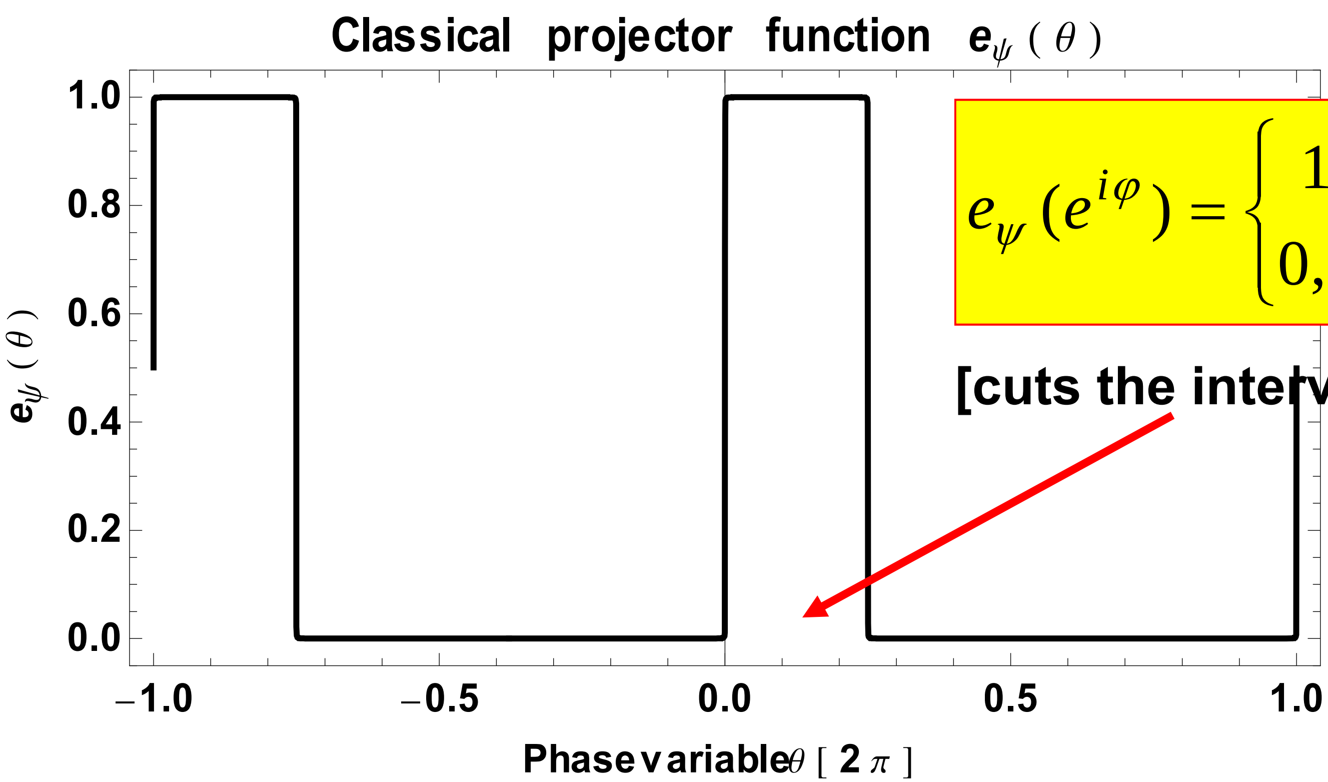
$$= \varphi_0 + \pi + \sum_{k=1}^{\infty} \frac{i}{k}[(e^{i(\varphi-\varphi_0)})^k - (e^{-i(\varphi-\varphi_0)})^k]$$

$$\Phi_{cl} = \varphi \quad (-\pi < \varphi < \pi)$$



Classical (2π-periodic) phase projector.[Cuts an interval.]

$$e_\psi(\varphi) = \sum_{k=1}^{\infty} \frac{i}{2\pi k}[(e^{-ik\psi} - 1)e^{ik\varphi} - (e^{+ik\psi} - 1)e^{-ik\varphi}]$$



$$e_\psi(e^{i\varphi}) = \begin{cases} 1, & (0 < \varphi < \psi) \\ 0, & (\psi < \varphi < 2\pi) \end{cases}$$

[cuts the interval (0,π/4).]

New phase operator and quantum 'projector'.

$$A = F\sqrt{N+\nu} \;\; (= E\sqrt{N}), \;\; \nu > 0$$

$$e^{i\varphi} \to F, \;\; e^{-i\varphi} \to F^{+}$$

$$\Phi_{\varphi_0} = \Phi_{\varphi_0}(F, F^{+})$$

[Strongly convergent in the domain where <(N+ν)^ν> is finite.]

$$= \varphi_0 + \pi + \sum_{k=1}^{\infty} \frac{i}{k}[F^{k} e^{-ik\varphi_0} - (F^{+})^{k} e^{+ik\varphi_0}]$$

$$\Phi_{\varphi_0} = \int_{\varphi_0}^{\varphi_0+2\pi} \psi dE_{\psi}$$

$$E_{\psi} = E_{\psi}(F, F^{+}) = \sum_{k=1}^{\infty} \frac{i}{2\pi k}[(e^{-ik\psi} - 1)F^{k} e^{-ik\varphi_0}$$

$$-(e^{+ik\psi} - 1)(F^{+})^{k} e^{+ik\varphi_0}]$$

[S. V., Regular phase operator and SU(1,1) coherent states of the harmonic oscillator. *Physica Scripta* 90 (7), 074053 (2015)]

Convergence in norm. [$\psi \in D(N)$ for $\nu = 2$]. The two infinite sums in Φ converge strongly.] Explicit form of the matrix elements of Φ.

$$F^k = f_k^{1/2}(N)E^k \quad (k \geq 0),$$

$$f_k(n) \equiv \frac{\Gamma(n+1+k)}{\Gamma(n+1)} \frac{\Gamma(n+\nu+1)}{\Gamma(n+\nu+1+k)}$$

$$\langle r|\Phi|s\rangle = \begin{cases} \pi, & r = s \\ \dfrac{i}{s-r} f_{|s-r|}^{1/2}(\min[s,r]), & r \neq s \end{cases}$$

$$\left\| F^k |\psi\rangle \right\|^2 < b_\nu \frac{1}{(k+\nu)^\nu} \sum_{n=0}^{\infty} (n+\nu)^\nu \, |c_n|^2$$

$$\frac{1}{k} \to \frac{1}{k^{1+\nu}}$$

Definition of the quantum phase probability distribution, and probability density distribution

$$\Phi = \int_{\varphi_0}^{\varphi_0+2\pi} \varphi \cdot dE_{\varphi}$$

$$\langle \Phi \rangle = \int_{\varphi_0}^{\varphi_0+2\pi} \varphi \cdot d[Tr(\hat{\rho} E_{\varphi})]$$

$$G(\varphi) \equiv Tr(\hat{\rho} E_{\varphi})$$

$$\Phi = \int_{\varphi_0}^{\varphi_0+2\pi} \varphi \cdot P_{\varphi} \cdot d\varphi$$

$$\langle \Phi \rangle = \int_{\varphi_0}^{\varphi_0+2\pi} \varphi \cdot Tr(\hat{\rho} P_{\varphi}) \cdot d\varphi$$

$$g(\varphi) \equiv Tr(\hat{\rho} P_{\varphi})$$

$$P_{\varphi_0} = (2\pi)^{-1} \sum_{k=-\infty}^{\infty} F_k e^{-ik\varphi_0}$$

We note that :

$$[N, \Phi] = i - 2\pi P_{\varphi_0}$$

[S. V., Regular phase operator and SU(1,1) coherent states of the harmonic oscillator. Physica Scripta 90 (7), 074053 (2015)]

The new phase distribution and the '<u>R – function</u>'. [An analogon of the well-known Q-function, which is Q(α)=<α| ρ |α>, with A|α>= α |α>.]

$$\Phi = \int_{\varphi_0}^{\varphi_0+2\pi} \varphi \cdot P_\varphi \cdot d\varphi$$

$$\langle \Phi \rangle = \int_{\varphi_0}^{\varphi_0+2\pi} \varphi \cdot Tr(\hat{\rho} P_\varphi) \cdot d\varphi$$

$$g(\varphi) \equiv Tr(\hat{\rho} P_\varphi) = \frac{\nu}{2\pi} \int_0^1 \frac{2rdr}{(1-r^2)^2} \int_0^{2\pi} d\theta \cdot p_\varphi(z) \cdot \langle z | \hat{\rho} | z \rangle$$

Poisson kernel :

$$p_\varphi(z) = \frac{1}{2\pi} \frac{1-r^2}{1-2r\cos(\varphi-\theta)+r^2}$$

R – function :

$$R(z) \equiv \langle z | \hat{\rho} | z \rangle$$

$$R(z) = R_{\hat{\rho}}(r,\theta)$$

[S. V., Regular phase operator and SU(1,1) coherent states of the harmonic oscillator. Physica Scripta 90 (7), 074053 (2015)]

Regular phase coherent states, diagonal representation of Φ.

$$F|z\rangle = z|z\rangle, \quad z \in \mathbb{D}.$$

SU(1,1) coherent states with Bargmann index κ, where $\nu = 2\kappa - 1$

$$|z\rangle = (1-|z|^2)^{\frac{1}{2}(\nu+1)} \sum_{n=0}^{\infty} \left[\frac{\Gamma(n+\nu+1)}{\Gamma(\nu+1)n!} \right]^{1/2} z^n |n\rangle$$

(Over)completeness: **Hyperbolic measure:**

$$\frac{\nu}{\pi} \int_{\mathbb{D}} d\mu(z)|z\rangle\langle z| = 1, \quad d\mu(z) \equiv (1-|z|^2)^{-2} dxdy$$

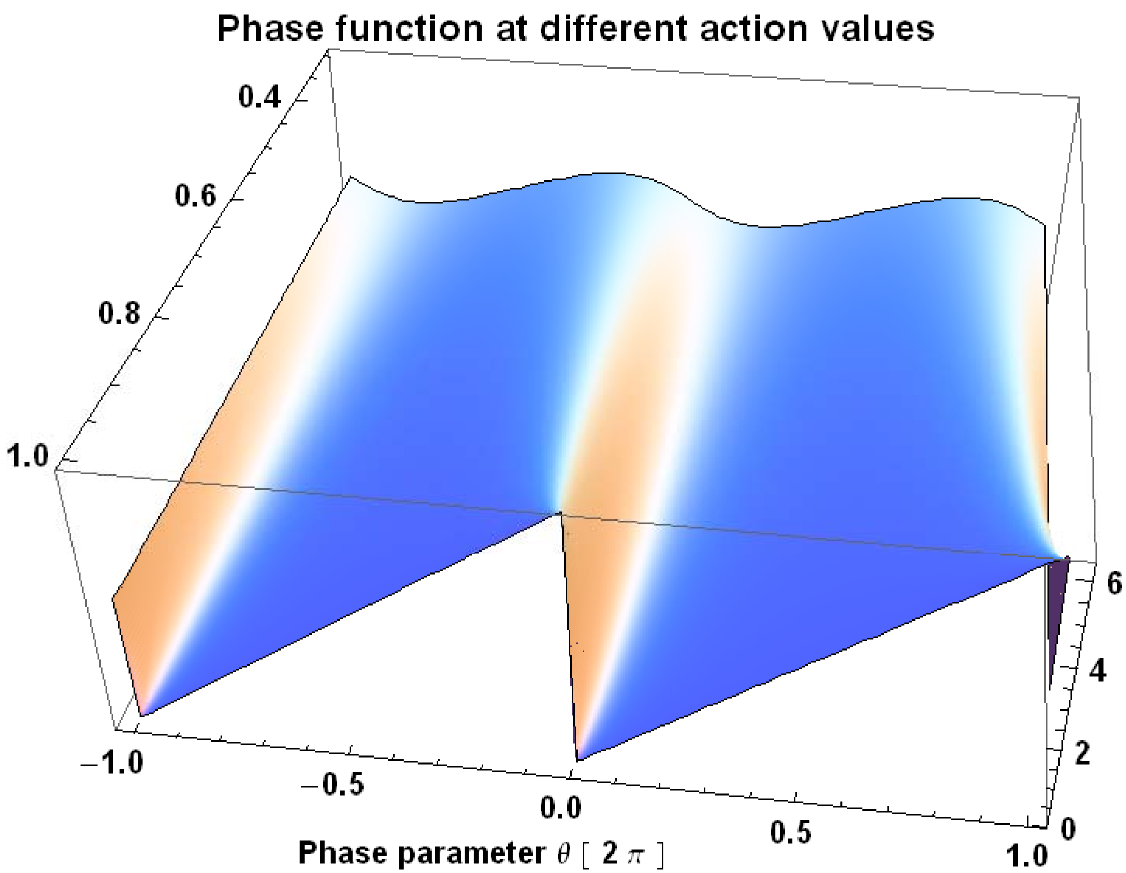
Blaschke function

Diagonal representation:

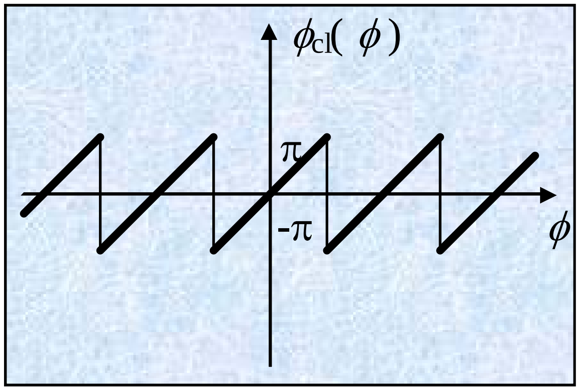
$$\Phi_{\varphi_0+\omega t} = \frac{\nu}{\pi} \int_{\mathbb{D}} d\mu(b)|b\rangle\varphi(b,t)\langle b|, \quad e^{i\varphi(b,t)} \equiv e^{i(\varphi_0+\pi)} \frac{e^{i\omega t} - b}{1 - b^* e^{i\omega t}}$$

[S. V., Regular phase operator and SU(1,1) coherent states of the harmonic oscillator. *Physica Scripta* 90 (7), 074053 (2015)]

The kernel of the diagonal representation of the new Φ.



[the classical phase function is also shown, for a comparison:]



[Fig. 2a of S. V, *Physica Scripta* 90 (7), 074053 (2015)]

The group SU(1,1) and the su(1,1) Lie algebra in the Holstein-Primakoff representation.

$$g = \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix}, \quad |\alpha|^2 - |\beta|^2 = 1$$

„The group SU(1,1) consist of all 2 x 2 unimodular matrices leaving invariant the Hermitian form $|z_1|^2$-$|z_2|^2$. Evidently, elements of SU(1,1) are parametrized with a pair of complex numbers." p.67.

$$[K_1, K_2] = -iK_0, \quad [K_2, K_0] = iK_1, \quad [K_0, K_1] = iK_2$$

$$K_1 = \tfrac{1}{2}(K_+ + K_-), \quad K_2 = \tfrac{1}{2i}(K_+ - K_-), \quad [K_-, K_+] = 2K_0$$

$$K_- = A\sqrt{N + \nu}, \quad K_+ = \sqrt{N + \nu}\, A^+, \quad K_0 = N + \kappa$$

$$[K_-, K_+] = 2K_0 = 2(N + \kappa)$$

κ is called the Bargmann index

$$K_0^2 - K_1^2 - K_2^2 = \kappa(\kappa - 1), \quad \kappa = \tfrac{1}{2}(\nu + 1)$$

SU(1,1) coherent states. Generation in the Holstein-Primakoff representation.

$$A = F\sqrt{N+\nu}$$

$$F|z\rangle = z|z\rangle, \quad z \in D.$$

$$|z\rangle = (1-|z|^2)^{\frac{1}{2}(\nu+1)} \sum_{n=0}^{\infty} \left[\frac{\Gamma(n+\nu+1)}{\Gamma(\nu+1)n!}\right]^{1/2} z^n |n\rangle$$

$$K_- = A\sqrt{N+\nu} = F(N+\nu)$$

$$|\kappa,\zeta\rangle = \exp(\xi K_+ - \xi^* K_-)|0\rangle = (1-|\zeta|^2)^{\kappa} \exp(\zeta K_+)|0\rangle$$

$$\xi = \rho e^{i\theta}, \quad \zeta = (\tanh\rho)e^{i\theta}$$

$$|\kappa,\zeta\rangle = (1-|\zeta|^2)^{\kappa} \sum_{n=0}^{\infty} \left[\frac{\Gamma(n+2\kappa)}{\Gamma(2\kappa)n!}\right]^{1/2} \zeta^n |n\rangle$$

Scalar product, phase kernel and the Blaschke functions.

$$\left|\langle z|b\rangle\right|^2 = \left[\frac{(1-|z|^2)(1-|b|^2)}{|1-z^*b|^2}\right]^{\nu+1} = \left[1-|B_a(z)|^2\right]^{\nu+1}$$

Blaschke function:

$$B_a(z) := \varepsilon\frac{z-b}{1-b^*z}$$

$$(z\in\mathbb{C}, a=(b,\varepsilon)\in\mathbb{B}:=\mathbb{D}\times\mathbb{T}, b^*z\neq 1)$$

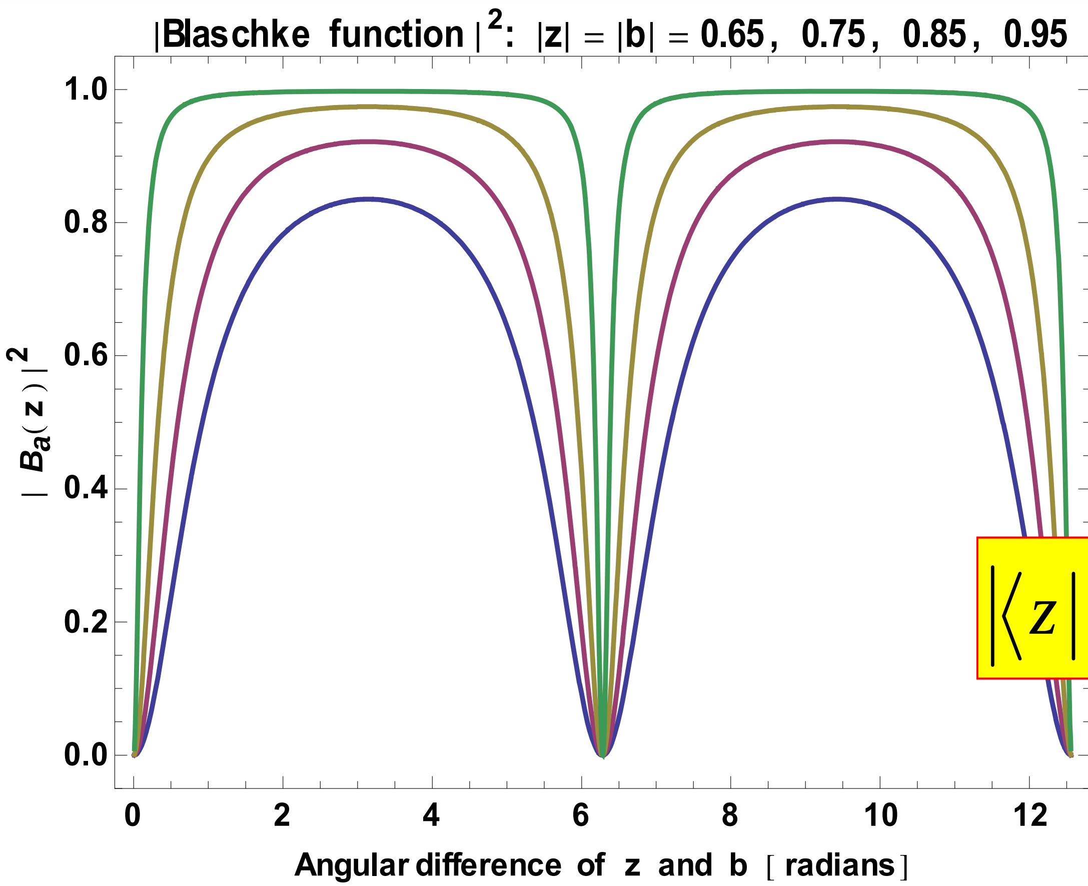
$$\mathbb{D}:=\{z\in\mathbb{C}:|z|<1\} \qquad \mathbb{T}:=\{z\in\mathbb{C}:|z|=1\}$$

The quantum phase function, i.e. the kernel of the diagonal representation of our Φ (in Heisenberg representation) is nothing else but the phase of the Blaschke function:

$$B_a(e^{it}) = e^{i\varphi(b,t)}$$

V S, The quantum phase of the photon as a Haar integral on the Blaschke group. Invited Talk S7.3.2 presented at Seminar 7: Quantum information and quantum computation, of LPHYS'16 [25th International Laser Physics Workshop, 11-15 July 2016., Yerevan, Armenia]

Angular dependence of the |Blaschke function|^2 at |z|=|b|.

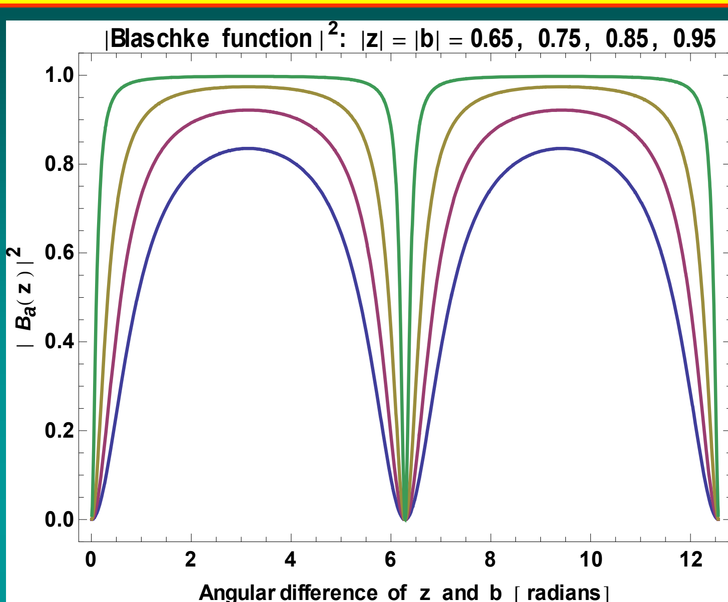


[we have seen that the scalar products of two regular phase coherent states are expressed as:]

$$\left|\langle z|b\rangle\right|^2 = \left(1 - | B_a(z) |^2\right)^{\nu+1}$$

V S, The quantum phase of the photon as a Haar integral on the Blaschke group. Invited Talk S7.3.2 presented at Seminar 7: Quantum information and quantum computation, of LPHYS'16 [25th International Laser Physics Workshop, 11-15 July 2016., Yerevan, Armenia]

Plots of angular dependence of the modulus squared of the Blaschke functions. Modulus squared of the scalar products of two regular phase coherent states.



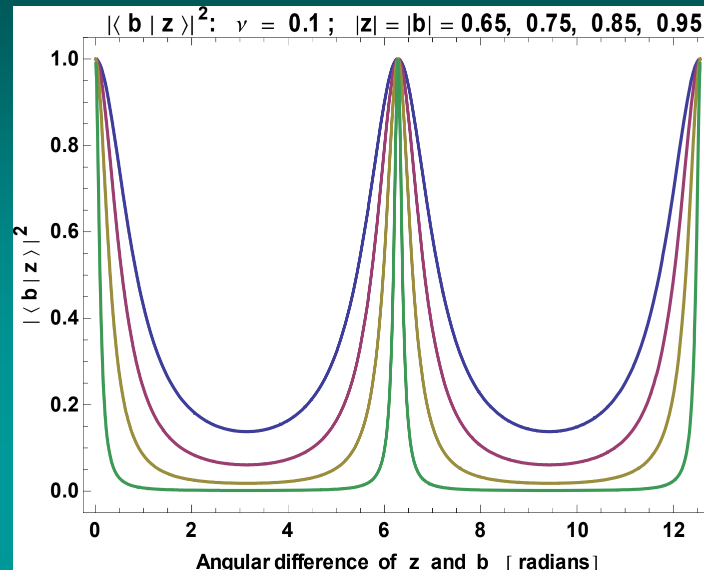
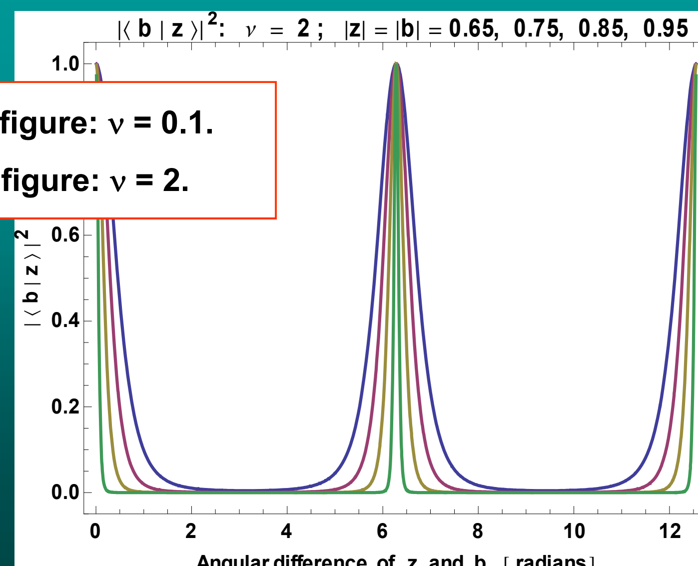


Figure 4. Shows the modulus squared of the Blaschke function, as a function of the angular differrence of and for four different |z|=|b | values. We have taken |z|= 0.65, 0.75, 0.85, 0.95, which correspond to the lowest, the two middle, and uppermost curves, respectively.



Upper figure: ν = 0.1.

Lower figure: ν = 2.

Illustration of the effect of the Blaschke group.

The function decomposition $(B_a \circ B_{a'})(z) := B_a(B_{a'}(z))$ induces a group $(B, \circ)$ on the parameter set $B := D \times T$.

$$a = (b, e^{it}) = (b_1 + ib_2, e^{it}) \in \mathbb{B} := \mathbb{D} \times \mathbb{T}$$

$$B_a(z) := e^{it} \frac{z - b}{1 - b^* z}$$

Figure copied from: Schipp Ferenc, Hyperbolic wavelets. To the memory of Prof. Mátyás Arató. [in Hungarian] *Alkalmazott Matematikai Lapok* 32, 1-40 (2015).

Brief summary on the Blaschke group.

The function decomposition $(B_a \circ B_{a'})(z) := B_a(B_{a'}(z))$ induces a group $(B,\circ)$ on the parameter set $B := D \times T$. The group action is defined as $B_a \circ B_{a'} = B_{a \circ a'}$. The $(B,\circ)$ is isomorphic with $(\{B_a, a \in B\}, \circ)$.

$$a := (b, \varepsilon) =: a_1 \circ a_2$$

$$b = \frac{b_1 \varepsilon_2^* + b_2}{1 + b_1 b_2^* \varepsilon_2^*} = B_{(-b_2, 1)}(b_1 \varepsilon_2^*)$$

$$a^{-1} = (-b\varepsilon, \varepsilon^*)$$

$$e := (0,1)$$

$$\varepsilon = \varepsilon_1 \frac{\varepsilon_2 + b_1 b_2^*}{1 + b_1^* b_2 \varepsilon_2} = B_{(-b_1 b_2^*, \varepsilon_1)}(\varepsilon_2)$$

The integral of some function *f* : B→C with respect to the left invariant Haar measure m on the group $(B,\circ)$ is expressed as:

$$\int_{\mathbb{B}} f(a)\,dm(a) = \frac{1}{2\pi} \int_{\mathbb{I}} \int_{\mathbb{D}} \frac{f(b, e^{it})}{(1 - |b|^2)^2} db_1 db_2 dt$$

$$\varepsilon = e^{it}$$

$$b = b_1 + i b_2$$

Varró S, The quantum phase of the photon a Haar integral on the Blaschke group. [Seminar 7 of LPHYS-16, 11-15 July 2016, Yerevan, Armenia] p. 13

Haar Alfréd

[1885. Budapest – 1933. Szeged]

Ist $f(X)$ eine beliebige, in der Gruppenmannigfaltigkeit $\mathfrak{G}$ definierte reelle oder komplexwertige Funktion von der Beschaffenheit, daß die Integrale

$$\int_{\mathfrak{G}} f(X)\, dX \quad \text{und} \quad \int_{\mathfrak{G}} |f(X)|^2\, dX$$

existieren, so gilt daher (für jedes Gruppenelement A) die Beziehung

$$(13) \quad \int_{\mathfrak{G}} f(XA)\, dX = \int_{\mathfrak{G}} f(X)\, dX, \qquad \int_{\mathfrak{G}} |f(XA)|^2\, dX = \int_{\mathfrak{G}} |f(X)|^2\, dX;$$

daher ist die Funktionaloperation O_A, die der Funktion $f(X)$ die Funktion $f(XA)$ zuordnet,

$$O_A(f(X)) = f(XA)$$

eine *orthogonale lineare Transformation.* Führt man in wohlbekannter Weise auf $\mathfrak{G}$ ein (vollständiges) orthogonales Funktionensystem ein, so gelangt man, indem man die Beziehungen zwischen den Fourierschen Koeffizienten der Funktionen $f(X)$ und $f(XA)$ aufstellt, zu *einem System von unendlichen orthogonalen Matrizen, das eine treu isomorphe Darstellung der vorgelegten Gruppe* $\mathfrak{G}$ *liefert.*

34. Haar Alfréd, Der Massbegriff in der Theorie der Kontinuierlichen Gruppen. *The Annals of Mathematics*, Second Series, Vol.34 No. 1 (Jan., 1933) pp 147 – 169.

Integral representation of the phase operator Φ.

The general left invariant Haar integral on the group (B,○).

$$\int_{\mathbb{B}} f(a)dm(a) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{\mathbb{D}} \frac{f(b,e^{it})}{(1-|b|^2)^2} db_1 db_2 dt$$

$$a = (b,e^{it}) = (b_1 + ib_2, e^{it}) \in \mathbb{B} := \mathbb{D}\times\mathbb{T}$$

The phase operator can also be expressed as a Haar integral of a positive operator (from the diagonal repre

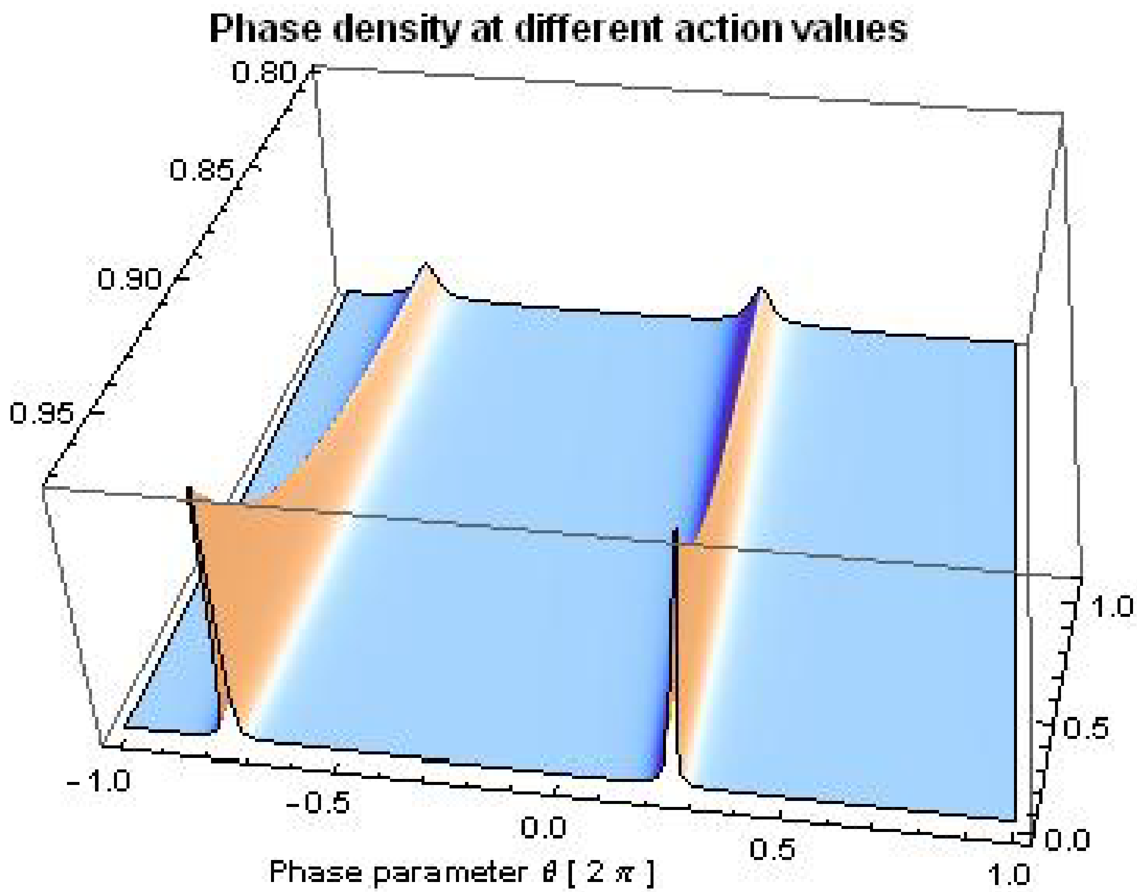
$$\hat{f}(a) := |b\rangle p(b,t)\langle b|$$

$$\Phi_{-\pi} = \frac{\nu}{\pi}\int_{-\pi}^{\pi} dt \int_{\mathbb{D}} \frac{db_1 db_2}{(1-|b|^2)^2} |b\rangle p(b,t)\langle b|$$

p(b,t) is the Poisson kernel (see figure on next page)

[Based on Eq. (5.1) in S. V, *Physica Scripta* 90 (7), 074053 (2015)]

The phase density function at different excitations



[Poisson kernel peaked around $\pi/4$ for different radial parameters (action values, or photon number expectation values (in the range $50 < N < 1000$).]

[Fig. 2b of S. V, *Physica Scripta* 90 (7), 074053 (2015)]

Heisenberg equation of motion for the physical phase.

$$i\hbar \frac{d\Phi_{\varphi_0}(t)}{dt} = [\Phi_{\varphi_0}(t), \hbar\omega(N+\tfrac{1}{2})]$$

$$[N, \Phi_{\varphi_0}] = i - 2\pi i P_{\varphi_0}$$

[Note that here N plays the role of the Hamiltonian, and NOT the role of the Action 'canonically conjugated' to the Angle (which is here the intial value of the phase).]

$$\Phi_{\varphi_0}(t) = \Phi_{\varphi_0}(0) - \omega t + 2\pi\omega\int_0^t P_{\varphi_0}(\tau)d\tau = \Phi_{dyn}(t) + \Phi_B(t)$$

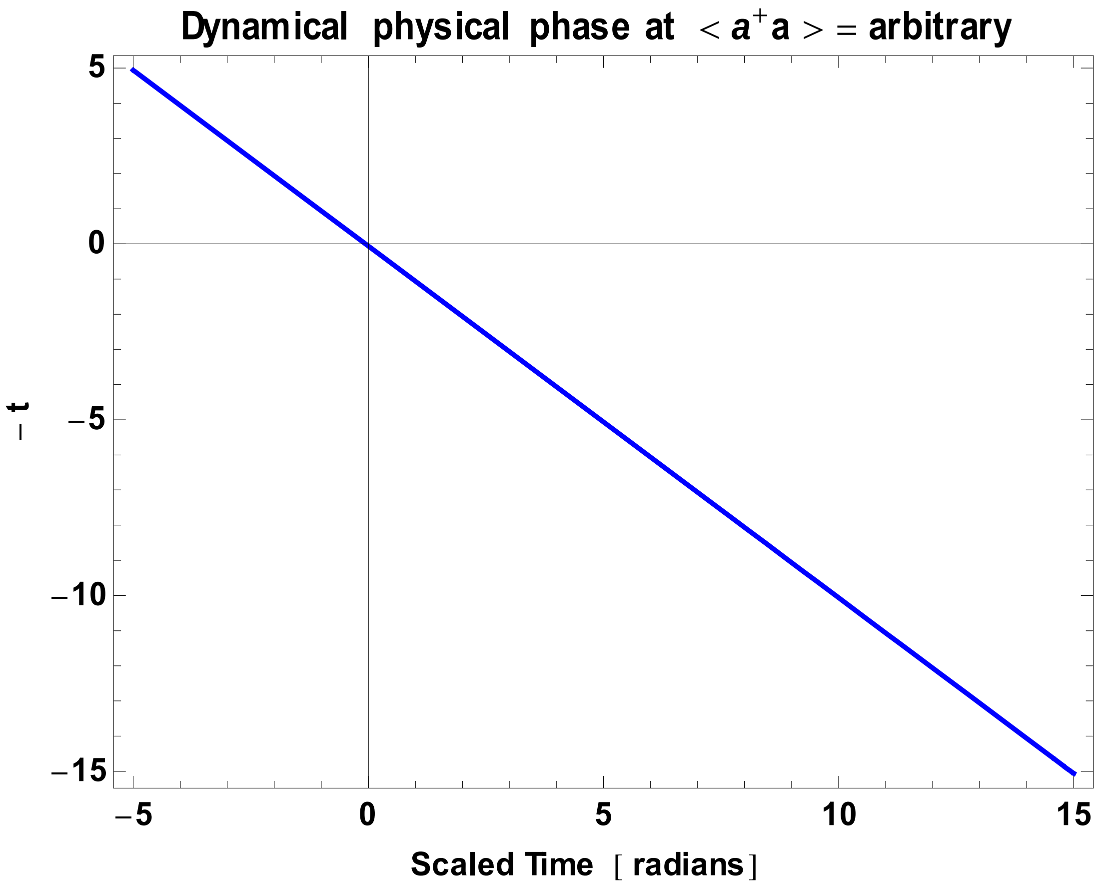
Dynamical part

Blaschke part

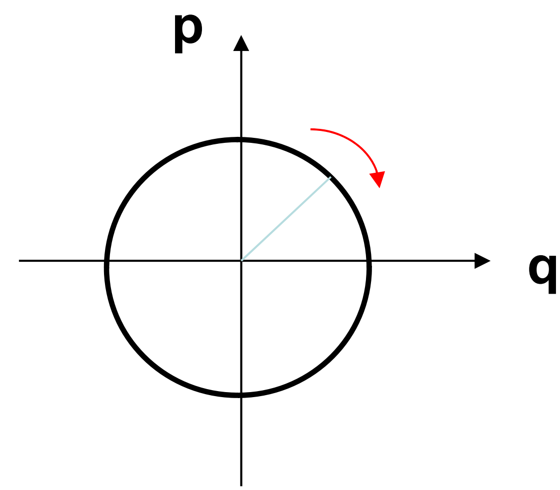
The solution contains a pure decrease –ωt (clock-wise rotation in phase-space), and a Blaschke contribution.

$$2\pi P_{\varphi_0}(\tau) = \mathrm{Re}\left\{\frac{1+Fe^{-i(\varphi_0+\omega\tau)}}{1-Fe^{-i(\varphi_0+\omega\tau)}}\right\}$$

Time-evolution of the dynamical phase; Φ_{dyn}.

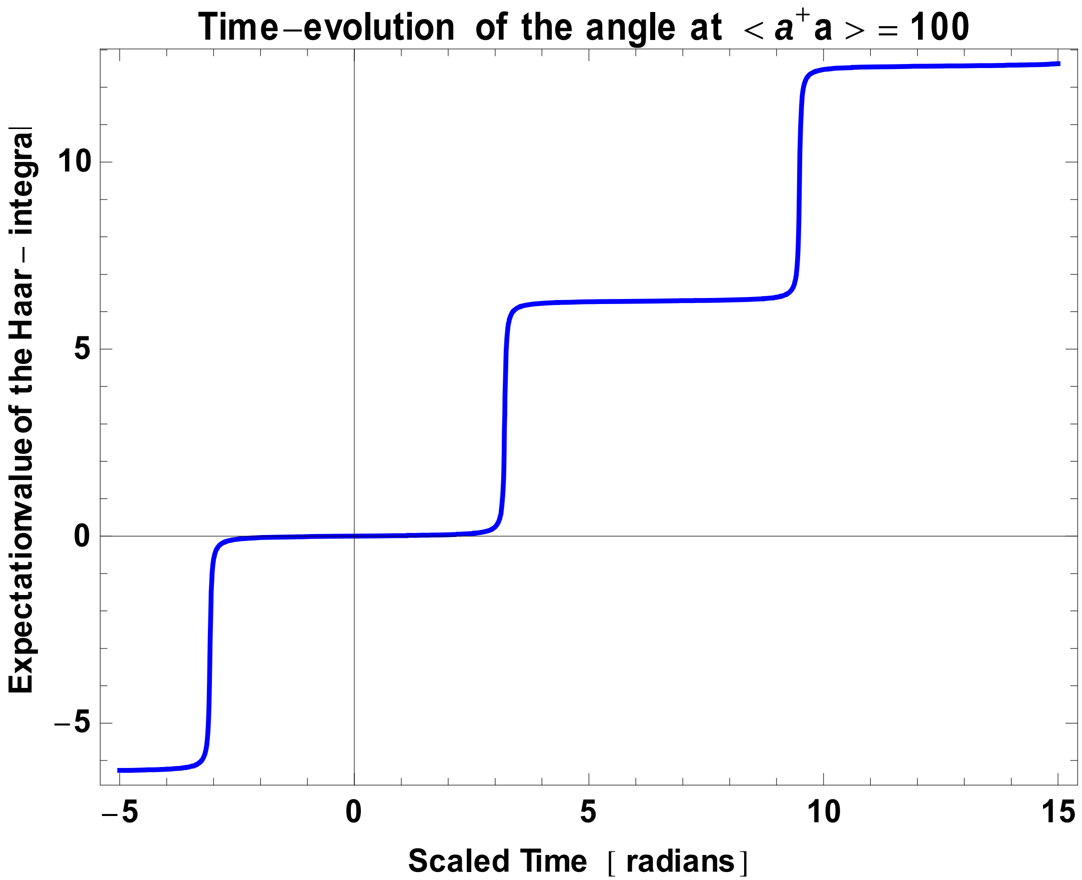


[This is the simple function $-\omega t$.]



{ Varró S, Invited talk at: 4th Work Meeting on Quantum Optics & Information [Regional Centre of the Hungarian Academy of Sciences at Pécs, 6-7 May 2016., Pécs, Hungary] }

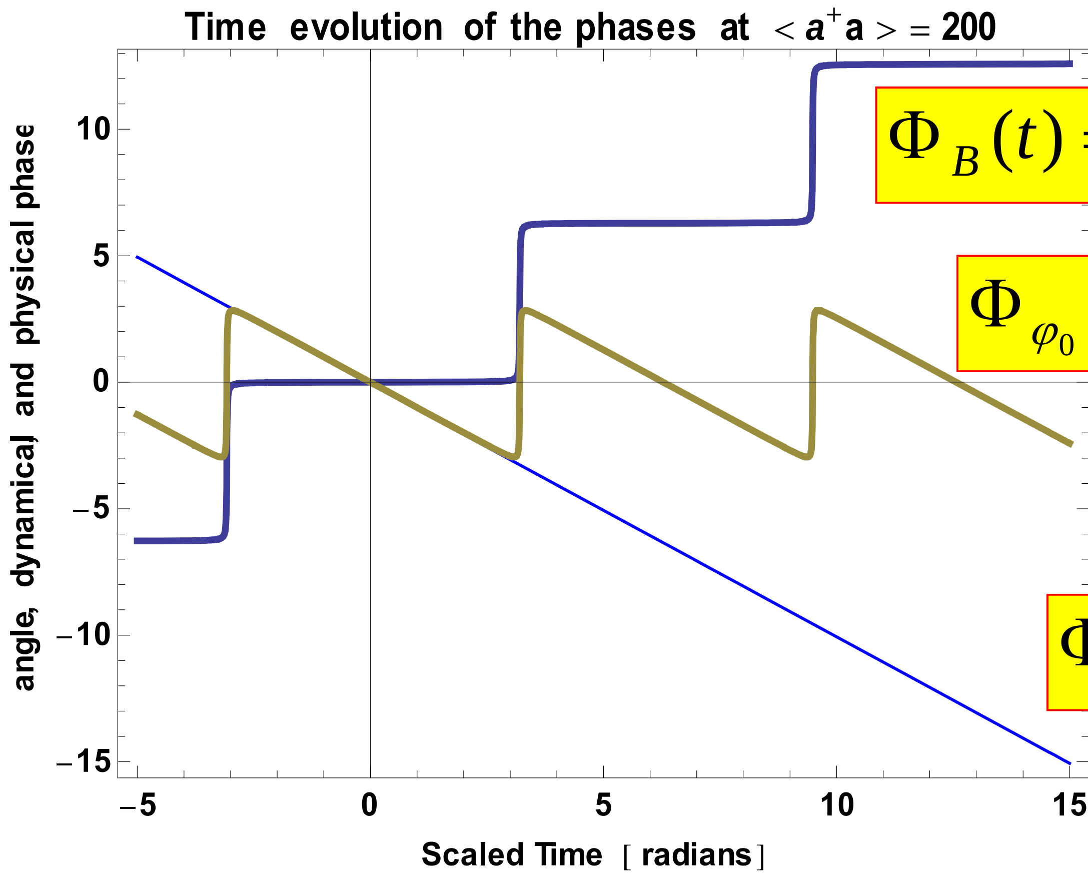
Time-evolution of the Blaschke part of the physical phase.



[This figure shows the time dependence of the expectation value of $\Phi_{\phi 0+\omega t}(0)$ in a phase coherent state, $|\zeta>$.]

{ Varró S, Invited talk at: 4th Work Meeting on Quantum Optics & Information [Regional Centre of the Hungarian Academy of Sciences at Pécs, 6-7 May 2016., Pécs, Hungary] }

Summary of the time evolution of the physical phase.



Unbounded

$$\Phi_B(t) = \Phi_{\varphi_0 + \omega t}(0) - \Phi_{\varphi_0}(0)$$

$$\Phi_{\varphi_0}(t) = \Phi_{dyn}(t) + \Phi_B(t)$$

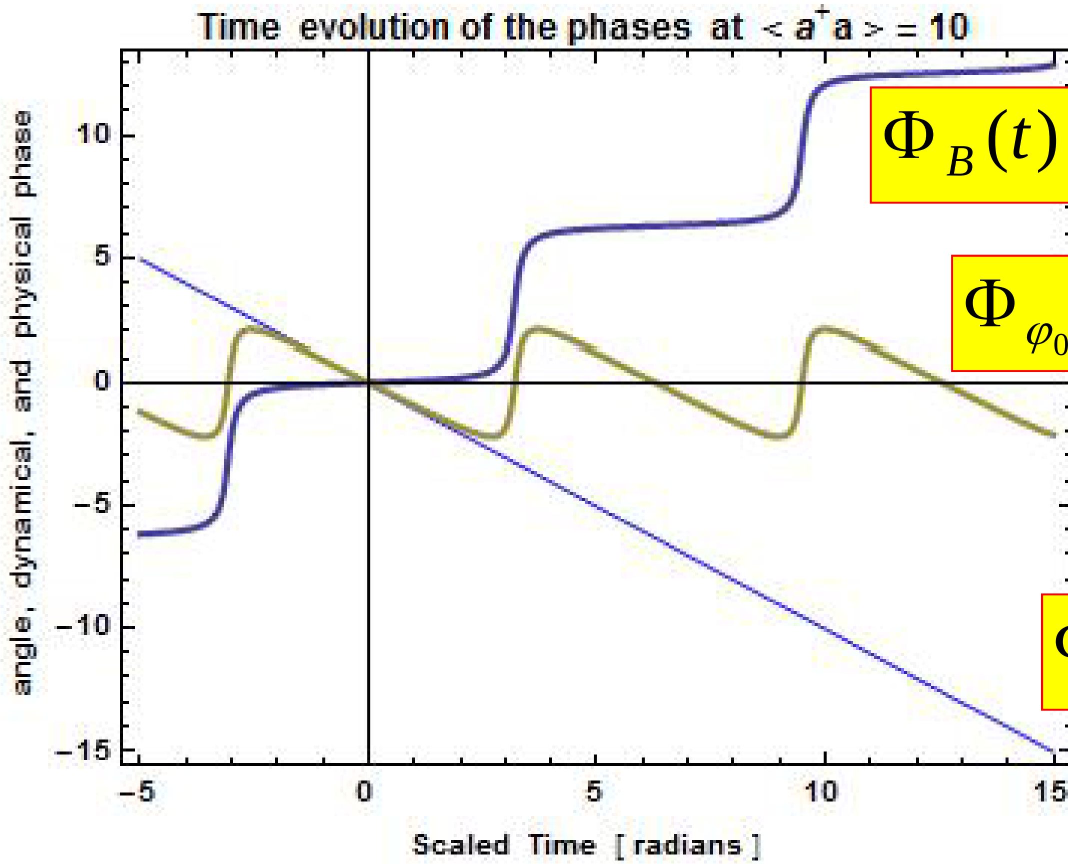
Sum: Periodic

$$\Phi_{dyn}(t) = \Phi_{\varphi_0}(0) - \omega t$$

Unbounded

V S, The quantum phase of the photon as a Haar integral on the Blaschke group. Invited Talk S7.3.2 presented at Seminar 7: Quantum information and quantum computation, of LPHYS'16 [25th International Laser Physics Workshop, 11-15 July 2016., Yerevan, Armenia]

Summary of the time evolution of the physical phase.



Unbounded

$$\Phi_B(t) = \Phi_{\varphi_0+\omega t}(0) - \Phi_{\varphi_0}(0)$$

$$\Phi_{\varphi_0}(t) = \Phi_{dyn}(t) + \Phi_B(t)$$

Sum: Periodic

$$\Phi_{dyn}(t) = \Phi_{\varphi_0}(0) - \omega t$$

Unbounded

V S, The quantum phase of the photon as a Haar integral on the Blaschke group. Invited Talk S7.3.2 presented at Seminar 7: Quantum information and quantum computation, of LPHYS'16 [25th International Laser Physics Workshop, 11-15 July 2016., Yerevan, Armenia]

Összefoglalás.

• **Áttekintettük a nemrég általunk bevezetett'reguláris fázisoperátor' és 'reguláris fázis koherens állapotok' (speciális SU(1,1) coherens állapotok) főbb tulajdonságait. Megmutattuk, hogy a fázis operátor a Blaschke-csoporton vett Haar-integrál. Bebizonyítottuk, hogy a teljes fizikai fázis periódikus.**

Acknowledgments.
I thank very illuminating and useful recent discussions on wavelet analysis and voice transforms with Dr. Margit Pap of University of Pécs, and Prof. Dr. Ferenc Schipp of the Eötvös Univesity Budapest. This work has been supported by the Hungarian Academy of Sciences, by the National Scientific Research Foundation OTKA, Grant No. K 104260, and, partially by the ELI-ALPS project . The ELI-ALPS project (GOP-1.1.1-12/B-2012-0001) is supported by the European Union and co-financed by the European Regional Development Fund.

To ‘Canonical references’.

Carruthers P and Nieto M M (1968): „Various pitfalls associated with the periodicity problem are avoided by employing periodic variables (sin ϕ and cos ϕ) to describe the phase variable."

REVIEWS OF MODERN PHYSICS VOLUME 40, NUMBER 2 APRIL 1968

Phase and Angle Variables in Quantum Mechanics*

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The quantum-mechanical description of phase and angle variables is reviewed, with emphasis on the proper mathematical description of these coordinates. The relations among the operators and state vectors under consideration are clarified in the context of the Heisenberg uncertainty relations. The familiar case of the azimuthal angle variable φ and its "conjugate" angular momentum L_z is discussed. Various pitfalls associated with the periodicity problem are avoided by employing periodic variables (sin φ and cos φ) to describe the phase variable. Well-defined uncertainty relations are derived and discussed. A detailed analysis of the three-dimensional harmonic oscillator excited in coherent states is given. A detailed analysis of the simple harmonic oscillator is given. The usual assumption that a (Hermitian) phase operator ϕ (conjugate to the number operator N) exists is shown to be erroneous. However, cosine and sine operators C and S exist and are the appropriate phase variables. A Poisson bracket argument using action-angle (rather J, cos ϕ, sin ϕ) variables is used to deduce C and S. The spectra and eigenfunctions of these operators are investigated, along with the important "phase-difference" periodic variables. The properties of the oscillator variables in the various types of states are analyzed with special attention to the uncertainty relations and the transition to the classical limit. The utility of coherent states as a basis for the description of the evolution of the density matrix is emphasized. In this basis it is easy to identify the classical Liouville equation in action-angle variables along with quantum-mechanical "corrections." Mention is made of possible physical applications to superfluid systems.

$$(\Delta N)^2(\Delta C)^2 \geq \frac{1}{4} < S >^2$$

$$(\Delta N)^2(\Delta S)^2 \geq \frac{1}{4} < C >^2$$

Carruthers P and Nieto M M, Phase and angle variables in quantum mechanics . *Reviews of Modern Physics* 40 (2), 411-440 (1968).

Canonical commutation relation for number and phase.

$$\mathscr{D}(N) = \{ f \in H^2 : \sum_{n=0}^{\infty} n^2 \,|\, f_n \,|^2 < \infty \}$$

$$\mathscr{C} = \{ f \in \mathscr{D}(N) : 0 = f(-1) = \sum_{n=0}^{\infty} (-1)^n f_n \}$$

$$h_m(z) = 1 + (-1)^{m+1} z^m \in \mathscr{C}$$

$$(h_m, f) = f_0 + (-1)^{m+1} f_m = 0 \rightarrow f \equiv 0$$

$\mathscr{C}$ is dense in H^2.

$$(g, [\Phi_{GW}, N]h) = \int \frac{d\theta}{2\pi} g^* \theta \frac{dh}{id\theta} - \int \frac{d\theta}{2\pi} \left(\frac{dg}{id\theta} \right)^* \theta h = i(g,h)$$

„Since $\mathscr{D}$(N) is dense in H^2, we can conclude that [Φ_{GW}, N]h = ih, for any h $\in$ $\mathscr{C}$. Thus (Φ,N) is a Heisenberg pair.”

Garrison J C and Wong J, Canonically conjugate pairs, uncertainty relations, and phase operators. *Journal of Mathematical Physics* 11 (8), 2242-2249 (1970).

Gantsog, Miranowicz, Tanas (1992): „There is, however, one important qualitative difference between the GW and PB formalism. The GW formalism introduces an anisotropy in to the phase distribution, and even the vacuum is anisotropic. This anisotropy is a consequence of their requirement that the number-phase commutator should be – i , i.e., the requirement that the number-phase operators are a Heisenberg pair."

PHYSICAL REVIEW A VOLUME 46, NUMBER 5 1 SEPTEMBER 1992

Phase properties of real field states: The Garrison-Wong versus Pegg-Barnett predictions

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(Received 24 March 1992)

A comparison is made of predictions for the phase variances and the phase distribution functions obtained from the Garrison-Wong and Pegg-Barnett formalisms for real field states that include number states, coherent states, and squeezed vacuum states. It is shown that both approaches lead to qualitatively different phase distributions. The Garrison-Wong approach predicts an anisotropy of the phase distribution that is inconsistent with the symmetry of the Wigner and Q functions.

PACS number(s): 42.50.Dv

I. INTRODUCTION

The problem of correct definition in quantum theory of an operator corresponding to the phase of a one-mode quantum field has a long history and has provoked many discussions and controversies. There have been numer-

glert [17] and Popov and Yarunin [18]. In both the latter papers one can find statements that the Pegg-Barnett phase operator is an "approximation" to the Garrison-Wong phase operator. Both approaches give the same results for highly excited states, but there are essential differences for the states with few photons. These dif-

Gantsog Ts, Miranowicz A and Tanas R, Phase properties of real field states: The Garrison-Wong versus Pegg-Barnett predictions. *Physical Review A* 46 (5), 2870-2876 (1992).

Gantsog, Miranowicz, Tanas (1992): „There is, however, one important qualitative difference between the GW and PB formalism. The GW formalism introduces an anisotropy in to the phase distribution, and even the vacuum is anisotropic. This anisotropy is a consequence of their requirement that the number-phase commutator should be – i , i.e., the requirement that the number-phase operators are a Heisenberg pair."

$$\left\langle f\left(\lim_{s\to\infty}\Phi_{PB}\right)\right\rangle \neq \lim_{s\to\infty}\left\langle f\left(\Phi_{PB}\right)\right\rangle$$

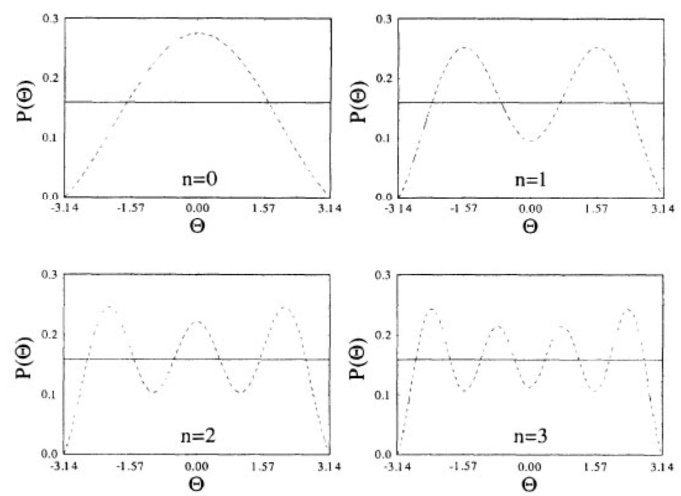

FIG. 4. Graphs of the phase distributions $P_{\mathrm{PB}}(\theta)$ (solid line) and $P_{\mathrm{GW}}(\theta)$ (dashed line) for the number states with $n = 0, 1, 2, 4$ in the rectangular coordinate system.

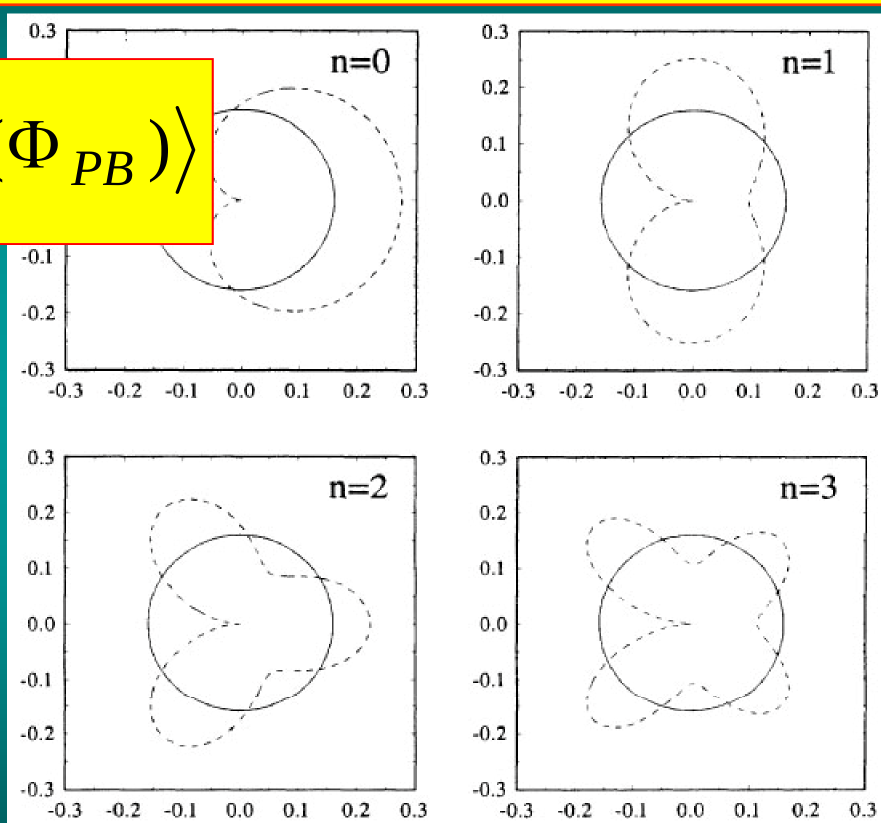

FIG. 5. Same as Fig. 4, but in the polar system.

Gantsog Ts, Miranowicz A and Tanas R, Phase properties of real field states: The Garrison-Wong versus Pegg-Barnett predictions. *Physical Review A* 46 (5), 2870-2876 (1992).