Megmaradó áramok konstrukciója mértékelméletekben rögzített háttér esetén

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The problem

Conserved currents for matter fields moving in fixed gravitational background:
$j^\mu = \sqrt{-g}\, T^{\mu\nu} h_\nu$, $T^{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta L_M}{\delta g_{\mu\nu}}$, $\partial_\mu j^\mu = 0$

Derivation: $\partial_\mu j^\mu = \sqrt{-g}\, \nabla_\mu (T^{\mu\nu} h_\nu) = \sqrt{-g} (\nabla_\mu T^{\mu\nu} h_\nu + T^{\mu\nu} \nabla_\mu h_\nu)$

Killing equation: $\nabla_\mu h_\nu + \nabla_\nu h_\mu = 0 \ \rightarrow \ T^{\mu\nu} \nabla_\mu h_\nu = 0$

divergencelessness of $T^{\mu\nu}$: $\nabla_\mu T^{\mu\nu} = 0 \rightarrow \ \nabla_\mu T^{\mu\nu} h_\nu = 0$

Difference from 'canonical' Noether current: $\partial_\nu \Sigma^{\mu\nu}$, $\Sigma^{\mu\nu} = -\Sigma^{\nu\mu}$

What can we do if there is also a fixed electromagnetic field?
$j^\mu = \sqrt{-g}\, (T^\mu{}_\nu h^\nu + h^\nu A_\nu \mathcal{J}^\mu)$, $\mathcal{J}^\mu = -\frac{1}{\sqrt{-g}} \frac{\delta L}{\delta A_\mu}$
generalized Lorentz law: $\nabla_\mu T^{\mu\nu} = F^{\nu\lambda} \mathcal{J}_\lambda$

- Generalization to arbitrary gauge theories?
- How is j^μ related to the 'canonical' Noether current?
- (How to handle half-integer spin matter fields?)

The Lagrangian density function

$$L(x^\mu, \chi_j, \partial_\mu \chi_j, \partial_{\mu\nu} \chi_j, \ldots, \Phi_i, \partial_\mu \Phi_i, \partial_{\mu\nu} \Phi_i, \ldots)$$

is a local function of Φ_i and χ_j. The action integral is $\int d^D x \, L$.

χ_j: 'fixed' fields

Φ_i: 'dynamical' fields

Auxiliary formula

Let G be some quantity of the form

$$G = \epsilon^{\alpha} G_{\alpha} + \partial_{\nu}\epsilon^{\alpha} G_{\alpha}^{\nu} + \partial_{\nu\lambda}\epsilon^{\alpha} G_{\alpha}^{\nu\lambda} + \dots$$

$G_{\alpha}^{\nu\lambda}$, $G_{\alpha}^{\nu\lambda\rho}$, ... are completely symmetric in the upper indices.
We have the formula

$$G = \epsilon^{\alpha} \hat{G}_{\alpha} + \partial_{\nu}\mathcal{G}^{\nu} \ ,$$

where

$$\hat{G}_{\alpha} = G_{\alpha} - \partial_{\nu} G_{\alpha}^{\nu} + \partial_{\nu\lambda} G_{\alpha}^{\nu\lambda} - \dots$$

and

$$\begin{aligned} \mathcal{G}^{\nu} \quad = \quad & \epsilon^{\alpha} G_{\alpha}^{\nu} + (\partial_{\lambda}\epsilon^{\alpha} G_{\alpha}^{\nu\lambda} - \epsilon^{\alpha}\partial_{\lambda} G_{\alpha}^{\nu\lambda}) \\ & + (\partial_{\lambda\rho}\epsilon^{\alpha} G_{\alpha}^{\nu\lambda\rho} - \partial_{\lambda}\epsilon^{\alpha}\partial_{\rho} G_{\alpha}^{\nu\lambda\rho} + \epsilon^{\alpha}\partial_{\lambda\rho} G_{\alpha}^{\nu\lambda\rho}) + \dots \end{aligned}$$

Derivation: $(uv)' = u'v + uv'$ ...

Noether's first theorem

A one-parameter transformation of the fields can be written after linearization in the parameter, denoted by s, as

$$\Phi_i \to \Phi_i + s\delta\Phi_i \ , \qquad\qquad \chi_j \to \chi_j + s\delta\chi_j$$

The associated first order variation of L is

$$\delta L = \frac{dL}{ds}|_{s=0} = \delta L_\Phi + \delta L_\chi$$

with

$$\begin{aligned}
\delta L_\Phi &= \frac{\partial L}{\partial \Phi_i}\delta\Phi_i + \frac{\partial L}{\partial(\partial_\mu \Phi_i)}\partial_\mu\delta\Phi_i + \frac{\partial L}{\partial(\partial_{\mu\nu}\Phi_i)}\partial_{\mu\nu}\delta\Phi_i + \dots \\
\delta L_\chi &= \frac{\partial L}{\partial \chi_j}\delta\chi_j + \frac{\partial L}{\partial(\partial_\mu \chi_j)}\partial_\mu\delta\chi_j + \frac{\partial L}{\partial(\partial_{\mu\nu}\chi_j)}\partial_{\mu\nu}\delta\chi_j + \dots
\end{aligned}$$

Applying the auxiliary formula with $\alpha \to i$, $\epsilon^\alpha \to \delta\Phi_i$, $G_\alpha^{\mu\nu\dots} \to \frac{\partial L}{\partial(\partial_{\mu\nu\dots}\Phi_i)}$ gives

$$\delta L_\Phi = \frac{\delta L}{\delta \Phi_i}\delta\Phi_i + \partial_\mu j^\mu_\Phi$$

Euler-Lagrange derivative:

$$\frac{\delta L}{\delta \Phi_i} = \frac{\partial L}{\partial \Phi_i} - \partial_\mu \frac{\partial L}{\partial(\partial_\mu \Phi_i)} + \partial_{\mu\nu} \frac{\partial L}{\partial(\partial_{\mu\nu} \Phi_i)} - \partial_{\mu\nu\lambda} \frac{\partial L}{\partial(\partial_{\mu\nu\lambda} \Phi_i)} + \dots$$

Euler-Lagrange current:

$$\begin{aligned} j_\Phi^\mu &= \frac{\partial L}{\partial(\partial_\mu \Phi_i)} \delta\Phi_i + \left(\frac{\partial L}{\partial(\partial_{\mu\nu} \Phi_i)} \partial_\nu \delta\Phi_i - \partial_\nu \frac{\partial L}{\partial(\partial_{\mu\nu} \Phi_i)} \delta\Phi_i \right) \\ &+ \left(\frac{\partial L}{\partial(\partial_{\mu\nu\lambda} \Phi_i)} \partial_{\nu\lambda} \delta\Phi_i - \partial_\nu \frac{\partial L}{\partial(\partial_{\mu\nu\lambda} \Phi_i)} \partial_\lambda \delta\Phi_i + \partial_{\nu\lambda} \frac{\partial L}{\partial(\partial_{\mu\nu\lambda} \Phi_i)} \delta\Phi_i \right) + \dots \end{aligned}$$

Symmetry transformation:

$$\delta\chi_j = 0, \qquad \delta L_\Phi = \partial_\mu K^\mu$$

$$\Rightarrow \quad \partial_\mu J_\Phi^\mu + \frac{\delta L}{\delta \Phi_i} \delta\Phi_i = 0$$

Noether current: $J_\Phi^\mu = j_\Phi^\mu - K^\mu$

(Euler-Lagrange equations: $\frac{\delta L}{\delta \Phi_i} = 0$)

Noether's second theorem

Gauge transformations (in general sense):

$$\begin{aligned} \delta\Phi_i &= \epsilon^\alpha \delta\Phi_{i\alpha} + \partial_\mu\epsilon^\alpha \, \delta\Phi^\mu_{i\alpha} + \partial_\mu\partial_\nu\epsilon^\alpha \, \delta\Phi^{\mu\nu}_{i\alpha} + \dots \\ \delta\chi_j &= \epsilon^\alpha \delta\chi_{j\alpha} + \partial_\mu\epsilon^\alpha \, \delta\chi^\mu_{j\alpha} + \partial_\mu\partial_\nu\epsilon^\alpha \, \delta\chi^{\mu\nu}_{j\alpha} + \dots \ , \end{aligned}$$

The auxiliary formula can be applied to $\frac{\delta L}{\delta\Phi_i}\delta\Phi_i$ and $\frac{\delta L}{\delta\chi_j}\delta\chi_j$:

$$\frac{\delta L}{\delta\Phi_i}\delta\Phi_i = \epsilon^\alpha B_{\Phi\alpha} + \partial_\mu\mathcal{B}^\mu_\Phi \qquad \frac{\delta L}{\delta\chi_j}\delta\chi_j = \epsilon^\alpha B_{\chi\alpha} + \partial_\mu\mathcal{B}^\mu_\chi \ ,$$

where

$$\begin{aligned} B_{\Phi\alpha} &= \frac{\delta L}{\delta\Phi_i}\delta\Phi_{i\alpha} - \partial_\mu\left(\frac{\delta L}{\delta\Phi_i}\delta\Phi^\mu_{i\alpha}\right) + \partial_\mu\partial_\nu\left(\frac{\delta L}{\delta\Phi_i}\delta\Phi^{\mu\nu}_{i\alpha}\right) - \dots \\ \mathcal{B}^\mu_\Phi &= \epsilon^\alpha\frac{\delta L}{\delta\Phi_i}\delta\Phi^\mu_{i\alpha} + \left[\partial_\nu\epsilon^\alpha\frac{\delta L}{\delta\Phi_i}\delta\Phi^{\mu\nu}_{i\alpha} - \epsilon^\alpha\partial_\nu\left(\frac{\delta L}{\delta\Phi_i}\delta\Phi^{\mu\nu}_{i\alpha}\right)\right] + \dots \end{aligned}$$

$B_{\Phi\alpha}$, $B_{\chi\alpha}$: Bianchi expressions

$\mathcal{B}^\mu_\Phi$, $\mathcal{B}^\mu_\chi$: Bianchi currents

$$\delta L = \delta L_\Phi + \delta L_\chi = \frac{\delta L}{\delta \Phi_i}\delta\Phi_i + \frac{\delta L}{\delta \chi_j}\delta\chi_j + \partial_\mu j^\mu_\Phi + \partial_\mu j^\mu_\chi$$

$$= \epsilon^\alpha B_{\Phi\alpha} + \partial_\mu \mathcal{B}^\mu_\Phi + \partial_\mu j^\mu_\Phi + \epsilon^\alpha B_{\chi\alpha} + \partial_\mu \mathcal{B}^\mu_\chi + \partial_\mu j^\mu_\chi$$

Gauge symmetry:

$$\delta L = \partial_\mu K^\mu$$

K^μ is assumed to be a homogeneous linear local function of ϵ^α:

$$K^\mu = \epsilon^\alpha K^\mu_\alpha + \partial_\nu \epsilon^\alpha K^{\mu\nu}_\alpha + \partial_{\nu\lambda}\epsilon^\alpha K^{\mu\nu\lambda}_\alpha + \ldots$$

$$\epsilon^\alpha (B_{\Phi\alpha} + B_{\chi\alpha}) = -\partial_\mu (\mathcal{B}^\mu_\Phi + \mathcal{B}^\mu_\chi + J^\mu) \; ,$$

where

$$J^\mu = j^\mu_\Phi + j^\mu_\chi - K^\mu$$

By Stokes' theorem

$$\int_\Omega d^D x \; \epsilon^\alpha (B_{\Phi\alpha} + B_{\chi\alpha}) = 0$$

Generalized Bianchi identity:

$$B_{\Phi\alpha} + B_{\chi\alpha} = 0$$

(For the Einstein-Hilbert Lagrangian: $\nabla_\mu G^{\mu\nu} = 0$; $G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R$)

Noether's third theorem

1.)

$$I^\mu = \mathcal{B}^\mu_\Phi + \mathcal{B}^\mu_\chi + J^\mu$$

is also conserved, regardless of the Euler-Lagrange equations. If Φ_i and χ_j satisfy their Euler-Lagrange equations, then $I^\mu = J^\mu$.

2.) I^μ is a homogeneous linear local function of ϵ^α and $\partial_\mu I^\mu = 0$ for arbitrary $\epsilon^\alpha \ \Rightarrow$

$$I^\mu = \partial_\nu \Sigma^{\mu\nu}$$

$\Sigma^{\mu\nu}$: superpotential, antisymmetric

$$\Sigma^{\mu\nu} = \frac{1}{2}\epsilon^\alpha(I^{\mu\nu}_\alpha - I^{\nu\mu}_\alpha) + \left[\frac{2}{3}\partial_\rho\epsilon^\alpha(I^{\mu\nu\rho}_\alpha - I^{\nu\mu\rho}_\alpha) - \frac{1}{3}\epsilon^\alpha\partial_\rho(I^{\mu\nu\rho}_\alpha - I^{\nu\mu\rho}_\alpha)\right]$$
$$+\left[\frac{3}{4}\partial_{\rho\lambda}\epsilon^\alpha(I^{\mu\nu\rho\lambda}_\alpha - I^{\nu\mu\rho\lambda}_\alpha) - \frac{2}{4}\partial_\rho\epsilon^\alpha\partial_\lambda(I^{\mu\nu\rho\lambda}_\alpha - I^{\nu\mu\rho\lambda}_\alpha)\right.$$
$$\left.+\frac{1}{4}\epsilon^\alpha\partial_{\rho\lambda}(I^{\mu\nu\rho\lambda}_\alpha - I^{\nu\mu\rho\lambda}_\alpha)\right] + \dots$$

$I^{\mu\nu\lambda\dots}_\alpha$:

$$I^\mu = \epsilon^\alpha I^\mu_\alpha + \partial_\nu\epsilon^\alpha I^{\mu\nu}_\alpha + \partial_{\nu\lambda}\epsilon^\alpha I^{\mu\nu\lambda}_\alpha + \dots$$

The construction

1.) Partial Bianchi identity:
If Φ_i satisfy their Euler-Lagrange equations, then $B_{\Phi\alpha} = 0$, thus

$$B_{\chi\alpha} = 0$$

without any assumption on χ_j. (Generalization of $\nabla_\mu T^{\mu\nu} = 0$.)

2.) Conservation of $\mathcal{B}^\mu_\chi$:
If $\delta\chi_j = 0$ for some particular ϵ^α, then $0 = \epsilon^\alpha B_{\chi\alpha} + \partial_\mu \mathcal{B}^\mu_\chi$. If, in addition, Φ_i satisfy their Euler-Lagrange equations, then $B_{\chi\alpha} = 0$, thus

$$0 = \partial_\mu \mathcal{B}^\mu_\chi$$

(Generalization of $\partial_\mu(\sqrt{-g}\, T^{\mu\nu} h_\nu) = 0$.)

3.) Relation between J^μ_Φ and $\mathcal{B}^\mu_\chi$:
If $\delta\chi_j = 0$, then $j^\mu_\chi = 0$, thus $J^\mu = J^\mu_\Phi$, and

$$-\mathcal{B}^\mu_\chi = J^\mu_\Phi + \mathcal{B}^\mu_\Phi - I^\mu = J^\mu_\Phi + \mathcal{B}^\mu_\Phi - \partial_\nu \Sigma^{\mu\nu}$$

Examples (Bianchi currents, partial Bianchi identities)

Matter fields propagating in fixed gravitational + Yang-Mills background

Elementary fields: $V_\mu^{\bar\mu}$, A_μ^a, matter fields $(\psi_{\check\nu}{}^{\check\rho}{}_{\alpha k})$

$g_{\mu\nu} = V_\mu^{\bar\mu} V_\nu^{\bar\nu} g_{\bar\mu\bar\nu}, \quad g_{\bar\mu\bar\nu} = \mathrm{diag}(1,-1,-1,-1)$

$\chi_j \equiv \{V_\mu^{\bar\mu}, A_\mu^a\}$, $\Phi_i \equiv \text{matter fields}$

Gauge group: diffeomorphisms + local Lorentz transformations + Yang-Mills gauge transformations

Covariant derivative for the complete gauge group:

$$D_\mu \psi_{\check\nu}{}^{\check\rho}{}_{\alpha k} = \nabla_\mu \psi_{\check\nu}{}^{\check\rho}{}_{\alpha k} + S_\alpha{}^\beta{}_\mu \psi_{\check\nu}{}^{\check\rho}{}_{\beta k} - \mathrm{i}\kappa A_\mu^a (t^a)_k{}^l \psi_{\check\nu}{}^{\check\rho}{}_{\alpha l}$$

∇_μ: Levi-Civita covariant differentiation

$S_\alpha{}^\beta{}_\mu = \frac{1}{2}(L^{\bar\nu\bar\lambda})_\alpha{}^\beta \mathcal{S}_{\bar\nu\bar\lambda\mu}$, $\mathcal{S}^{\bar\lambda}{}_{\bar\eta\mu} = -V_{\bar\eta}^\nu \nabla_\mu V_\nu^{\bar\lambda}$

$(L^{\bar\nu\bar\lambda})_\alpha{}^\beta$: generators of the Lorentz group

$(t^a)_k{}^l$: generators of the Yang-Mills gauge group

$\check\nu \equiv \nu_1\nu_2\ldots\nu_n$, $\check\rho \equiv \rho_1\rho_2\ldots\rho_m$

1. Yang-Mills gauge symmetry:

$$\delta V_\mu^{\bar{\mu}} = 0, \qquad \delta A_\mu^a = D_\mu \epsilon^a$$

Partial Bianchi identity:

$$B_\chi^a = \sqrt{-g} D_\mu \mathcal{J}^{\mu a} = 0, \qquad \mathcal{J}^{\mu a} = -\frac{1}{\sqrt{-g}} \frac{\delta L}{\delta A_\mu^a}$$

Bianchi current:

$$\mathcal{B}_\chi^\mu = -\sqrt{-g}\, \epsilon^a \mathcal{J}^{\mu a}$$

$\mathcal{B}_\chi^\mu$ is conserved if $\delta A_\mu^a = D_\mu \epsilon^a = 0$, and the matter fields satisfy their Euler-Lagrange equations.

Derivation: $\partial_\mu \mathcal{B}_\chi^\mu = -\sqrt{-g}\, \nabla_\mu (\epsilon^a \mathcal{J}^{\mu a}) = -\sqrt{-g}\, (D_\mu \epsilon^a \mathcal{J}^{\mu a} + \epsilon^a D_\mu \mathcal{J}^{\mu a})$

2. Local Lorentz symmetry:

$$\delta V^{\bar{\mu}}_{\mu} = -\omega^{\bar{\lambda}\bar{\rho}}(L_{\bar{\lambda}\bar{\rho}})_{\bar{\nu}}{}^{\bar{\mu}} V^{\bar{\nu}}_{\mu}, \qquad (\delta g_{\mu\nu} = 0), \qquad \delta A^{a}_{\mu} = 0$$

Partial Bianchi identity:

$$B_{\chi,\bar{\lambda}\bar{\rho}} = -2\sqrt{-g}\,\mathbb{T}_{\bar{\lambda}\bar{\rho}} = 0$$

where $\mathbb{T}_{\bar{\lambda}\bar{\rho}}$ is the antisymmetric part in the decomposition

$$V_{\bar{\lambda}\rho}\frac{\delta L}{\delta V^{\bar{\rho}}_{\rho}} = \sqrt{-g}(-T_{\bar{\lambda}\bar{\rho}} + \mathbb{T}_{\bar{\lambda}\bar{\rho}})$$

Bianchi current: $\mathcal{B}^{\mu}_{\chi} = 0$

3. Diffeomorphism symmetry:

$$\delta V_\mu^{\bar\mu} = -h^\nu \nabla_\nu V_\mu^{\bar\mu} - \nabla_\mu h^\nu V_\nu^{\bar\mu}, \qquad \delta A_\mu^a = -h^\nu \nabla_\nu A_\mu^a - \nabla_\mu h^\nu A_\nu^a$$

$$(\delta g_{\mu\nu} = -\nabla_\mu h_\nu - \nabla_\nu h_\mu)$$

Partial Bianchi identity:

$$\begin{aligned} B_{\chi\mu} \quad &= \quad \sqrt{-g}\,(-\nabla_\nu T^\nu{}_\mu + \nabla_\nu \mathbb{T}^\nu{}_\mu + \mathbb{T}^{\lambda\nu} V_{\lambda\bar\mu} \nabla_\mu V_\nu^{\bar\mu} \\ &\quad -\mathcal{J}^{\nu a} F_{\nu\mu}^a - D_\nu \mathcal{J}^{\nu a} A_\mu^a) = 0 \end{aligned}$$

or

$$B_{\chi\mu} = -\sqrt{-g}\,(\nabla_\nu T^\nu{}_\mu + \mathcal{J}^{\nu a} F_{\nu\mu}^a) = 0$$

Bianchi current:

$$\mathcal{B}_\chi^\mu = \sqrt{-g}(T^\mu{}_\nu - \mathbb{T}^\mu{}_\nu + A_\nu^a \mathcal{J}^{\mu a}) h^\nu$$

or

$$\mathcal{B}_\chi^\mu = \sqrt{-g}(T^\mu{}_\nu + A_\nu^a \mathcal{J}^{\mu a}) h^\nu$$

Derivation of $\partial_\mu \mathcal{B}^\mu_\chi = 0$:

$$\partial_\mu \mathcal{B}^\mu_\chi = \sqrt{-g}(\nabla_\mu T^\mu{}_\nu h^\nu + T^\mu{}_\nu \nabla_\mu h^\nu + D_\mu \mathcal{J}^{\mu a} A^a_\nu h^\nu + \mathcal{J}^{\mu a} D_\mu A^a_\nu h^\nu$$
$$+ \mathcal{J}^{\mu a} A^a_\nu \nabla_\mu h^\nu)$$

$$T^\mu{}_\nu \nabla_\mu h^\nu = 0$$

$$D_\mu \mathcal{J}^{\mu a} A^a_\nu h^\nu = 0$$

$$\nabla_\mu T^\mu{}_\nu h^\nu \;\rightarrow\; -\mathcal{J}^{\mu a} F^a_{\mu\nu} h^\nu$$

$$\Rightarrow \partial_\mu \mathcal{B}^\mu_\chi = \sqrt{-g}(-\mathcal{J}^{\mu a} F^a_{\mu\nu} h^\nu + \mathcal{J}^{\mu a} D_\mu A^a_\nu h^\nu + \mathcal{J}^{\mu a} A^a_\nu \nabla_\mu h^\nu)$$

Using $h^\mu \nabla_\mu A^a_\nu + \nabla_\nu h^\mu A^a_\mu = 0$, $\mathcal{J}^{\mu a} A^a_\nu \nabla_\mu h^\nu \;\rightarrow\; -\mathcal{J}^{\mu a} \nabla_\nu A^a_\mu h^\nu$, thus

$$\partial_\mu \mathcal{B}^\mu = \sqrt{-g}(-\mathcal{J}^{\mu a} F^a_{\mu\nu} h^\nu + \mathcal{J}^{\mu a} D_\mu A^a_\nu h^\nu - \mathcal{J}^{\mu a} \nabla_\nu A^a_\mu h^\nu).$$

This is zero, since $F^a_{\mu\nu} = D_\mu A^a_\nu - \nabla_\nu A^a_\mu$.

4. 'Mixed' symmetry:
joint infinitesimal diffeomorphism and Yang-Mills gauge transformation corresponding to (h^μ, ϵ^a):

$$\delta A^a_\nu = -h^\mu \nabla_\mu A^a_\nu - \nabla_\nu h^\mu A^a_\mu + D_\nu \epsilon^a$$

Bianchi current:

$$\mathcal{B}^\mu_\chi = \sqrt{-g}\,[T^\mu{}_\nu h^\nu + (h^\nu A^a_\nu - \epsilon^a)\mathcal{J}^{\mu a}]$$

5. If the Yang-Mills gauge symmetry of the Lagrangian is not required and A^a_μ are merely some fixed covector fields, then $D_\nu \mathcal{J}^{\nu a} = 0$ does not necessarily hold, and one has the partial Bianchi identity

$$B_{\chi\mu} = \sqrt{-g}(-\nabla_\nu T^\nu{}_\mu - \mathcal{J}^{\nu a} F^a_{\nu\mu} - D_\nu \mathcal{J}^{\nu a} A^a_\mu) = 0$$

where D_μ and $F^a_{\nu\mu}$ are defined in the same way as in abelian gauge theory. The Bianchi current for a diffeomorphism symmetry of the metric and A^a_μ remains unchanged.

6. Fixed fields: the metric (or the tetrad) and a real scalar field ϕ

$$\delta\phi = -h^{\mu}\partial_{\mu}\phi$$

Partial Bianchi identity (using $\mathbb{T}_{\bar{\mu}\bar{\nu}} = 0$):

$$B_{\chi\mu} = \sqrt{-g}(-\nabla_{\nu} T^{\nu}{}_{\mu} - \mathcal{J}_{\phi}\partial_{\mu}\phi) = 0 \qquad \mathcal{J}_{\phi} = \frac{1}{\sqrt{-g}}\frac{\delta L}{\delta\phi}$$

Bianchi current:

$$\mathcal{B}^{\mu}_{\chi} = \sqrt{-g}\, T^{\mu}{}_{\nu}\, h^{\nu}$$

Derivation of $\partial_{\mu}\mathcal{B}^{\mu}_{\chi} = 0$: $\nabla_{\mu}(T^{\mu}{}_{\nu}\, h^{\nu}) = \nabla_{\mu} T^{\mu}{}_{\nu}\, h^{\nu} + T^{\mu}{}_{\nu}\nabla_{\mu} h^{\nu}$, second term is zero in virtue of the Killing equation, the first term can be rewritten as $\nabla_{\mu} T^{\mu}{}_{\nu}\, h^{\nu} = -\mathcal{J}_{\phi} h^{\nu}\partial_{\nu}\phi$

7. If only the scalar field is fixed, then

$$B_{\chi\mu} = \sqrt{-g}(-\mathcal{J}_{\phi}\partial_{\mu}\phi) = 0$$

$$\mathcal{B}^{\mu}_{\chi} = 0$$

Superpotentials

$$L = \sqrt{-g}\,\hat{L}\big(V^{\bar{\mu}}_{\mu}, \psi_{\check{\nu}}{}^{\check{\rho}}{}_{\alpha k}, \bar{\psi}_{\check{\nu}}{}^{\check{\rho}\alpha k}, D_{\mu}\psi_{\check{\nu}}{}^{\check{\rho}}{}_{\alpha k}, D_{\mu}\bar{\psi}_{\check{\nu}}{}^{\check{\rho}\alpha k}\big)$$

Irreducible real representations of the Lorentz group: $(\frac{l_1}{2}, \frac{l_2}{2}) \oplus (\frac{l_2}{2}, \frac{l_1}{2})$, $l_1 \neq l_2$, $l_1, l_2 \in \mathbb{N}$, and $(\frac{l}{2}, \frac{l}{2})$
The simplest examples: Dirac spinor representation $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$, Minkowski representation $(\frac{1}{2}, \frac{1}{2})$

Under the Yang-Mills gauge group $\psi_{\check{\nu}}{}^{\check{\rho}}{}_{\alpha k}$ transforms according to a not necessarily irreducible finite dimensional unitary representation; the corresponding index is denoted by k.

$\hat{L}$ is a scalar function:
$K^{\mu} = 0$ for Yang-Mills gauge transformations and local Lorentz transformations,

$K^{\mu} = -h^{\mu} L$ for diffeomorphisms

The first order variation of the matter field under a Yang-Mills gauge transformation parametrized by ϵ^a is

$$\delta\psi_{\check{\nu}}{}^{\check{\rho}}{}_{\alpha k} = \mathrm{i}\kappa\epsilon^a (t^a)_k{}^l \psi_{\check{\nu}}{}^{\check{\rho}}{}_{\alpha l}$$

The first order variation of the matter field under a local Lorentz transformation is

$$\delta\psi_{\check{\nu}}{}^{\check{\rho}}{}_{\alpha k} = \omega^{\bar{\mu}\bar{\nu}} (L_{\bar{\mu}\bar{\nu}})_\alpha{}^\beta \psi_{\check{\nu}}{}^{\check{\rho}}{}_{\beta k}$$

The first order variation of the matter field under a diffeomorphism generated by h^μ is

$$\begin{aligned} \delta\psi_{\check{\nu}}{}^{\check{\rho}}{}_{\alpha k} \quad &= \quad -h^\mu \nabla_\mu \psi_{\check{\nu}}{}^{\check{\rho}}{}_{\alpha k} \\ &-\nabla_{\nu_1} h^\mu \psi_{\mu\nu_2\ldots\nu_n}{}^{\check{\rho}}{}_{\alpha k} - \nabla_{\nu_2} h^\mu \psi_{\nu_1\mu\nu_3\ldots\nu_n}{}^{\check{\rho}}{}_{\alpha k} - \cdots - \nabla_{\nu_n} h^\mu \psi_{\nu_1\ldots\nu_{n-1}\mu}{}^{\check{\rho}}{}_{\alpha k} \\ &+\nabla_\mu h^{\rho_1} \psi_{\check{\nu}}{}^{\mu\rho_2\ldots\rho_m}{}_{\alpha k} + \nabla_\mu h^{\rho_2} \psi_{\check{\nu}}{}^{\rho_1\mu\rho_3\ldots\rho_m}{}_{\alpha k} + \cdots + \nabla_\mu h^{\rho_m} \psi_{\check{\nu}}{}^{\rho_1\ldots\rho_{m-1}\mu}{}_{\alpha k} \end{aligned}$$

Let's introduce the notation

$$P^{\mu\check{\nu}}{}_{\check{\rho}}{}^{\alpha k} = \frac{\partial \hat{L}}{\partial D_\mu \psi_{\check{\nu}}{}^{\check{\rho}}{}_{\alpha k}} \qquad \bar{P}^{\mu\check{\nu}}{}_{\check{\rho}\alpha k} = \frac{\partial \hat{L}}{\partial D_\mu \bar{\psi}_{\check{\nu}}{}^{\check{\rho}\alpha k}}$$

$$\mathbb{Q}^{\mu\bar{\delta}\bar{\lambda}} = \frac{1}{2} P^{\mu\check{\nu}}{}_{\check{\rho}}{}^{\alpha k} \psi_{\check{\nu}}{}^{\check{\rho}}{}_{\beta k} (L^{\bar{\delta}\bar{\lambda}})_\alpha{}^\beta - \frac{1}{2} \bar{P}^{\mu\check{\nu}}{}_{\check{\rho}\alpha k} \bar{\psi}_{\check{\nu}}{}^{\check{\rho}\beta k} (L^{\bar{\delta}\bar{\lambda}})_\beta{}^\alpha$$

$$
\begin{aligned}
I^{\mu} \quad = \quad & \sqrt{-g}\Big[\nabla_{\nu}[2(\mathbb{Q}^{(\mu\lambda)\nu}h_{\lambda} - \mathbb{Q}^{(\nu\lambda)\mu}h_{\lambda}) - \mathbb{Q}^{\lambda\mu\nu}h_{\lambda}] \\
& -\frac{1}{2}\sum_{i=1}^{n}\nabla_{\nu}[P^{\mu\ldots\nu\ldots}{}_{\check{\rho}}{}^{\alpha k}\psi_{\ldots}{}^{\lambda}{}_{\ldots}{}^{\check{\rho}}{}_{\alpha k}h_{\lambda} - P^{\nu\ldots\mu\ldots}{}_{\check{\rho}}{}^{\alpha k}\psi_{\ldots}{}^{\lambda}{}_{\ldots}{}^{\check{\rho}}{}_{\alpha k}h_{\lambda}] \\
& +\frac{1}{2}\sum_{i=1}^{n}\nabla_{\nu}[P^{\mu\ldots\lambda\ldots}{}_{\check{\rho}}{}^{\alpha k}\psi_{\ldots}{}^{\nu}{}_{\ldots}{}^{\check{\rho}}{}_{\alpha k}h_{\lambda} - P^{\nu\ldots\lambda\ldots}{}_{\check{\rho}}{}^{\alpha k}\psi_{\ldots}{}^{\mu}{}_{\ldots}{}^{\check{\rho}}{}_{\alpha k}h_{\lambda}] \\
& +\frac{1}{2}\sum_{i=1}^{n}\nabla_{\nu}[P^{\lambda\ldots\mu\ldots}{}_{\check{\rho}}{}^{\alpha k}\psi_{\ldots}{}^{\nu}{}_{\ldots}{}^{\check{\rho}}{}_{\alpha k}h_{\lambda} - P^{\lambda\ldots\nu\ldots}{}_{\check{\rho}}{}^{\alpha k}\psi_{\ldots}{}^{\mu}{}_{\ldots}{}^{\check{\rho}}{}_{\alpha k}h_{\lambda}] \\
& -\frac{1}{2}\sum_{i=1}^{m}\nabla_{\rho}[P^{\mu\check{\nu}}{}_{\ldots}{}^{\rho}{}_{\ldots}{}^{\alpha k}\psi_{\check{\nu}}{}^{\ldots\lambda\ldots}{}_{\alpha k}h_{\lambda} - P^{\rho\check{\nu}}{}_{\ldots}{}^{\mu}{}_{\ldots}{}^{\alpha k}\psi_{\check{\nu}}{}^{\ldots\lambda\ldots}{}_{\alpha k}h_{\lambda}] \\
& +\frac{1}{2}\sum_{i=1}^{m}\nabla_{\rho}[P^{\mu\check{\nu}}{}_{\ldots}{}^{\lambda}{}_{\ldots}{}^{\alpha k}\psi_{\check{\nu}}{}^{\ldots\rho\ldots}{}_{\alpha k}h_{\lambda} - P^{\rho\check{\nu}}{}_{\ldots}{}^{\lambda}{}_{\ldots}{}^{\alpha k}\psi_{\check{\nu}}{}^{\ldots\mu\ldots}{}_{\alpha k}h_{\lambda}] \\
& +\frac{1}{2}\sum_{i=1}^{m}\nabla_{\rho}[P^{\lambda\check{\nu}}{}_{\ldots}{}^{\mu}{}_{\ldots}{}^{\alpha k}\psi_{\check{\nu}}{}^{\ldots\rho\ldots}{}_{\alpha k}h_{\lambda} - P^{\lambda\check{\nu}}{}_{\ldots}{}^{\rho}{}_{\ldots}{}^{\alpha k}\psi_{\check{\nu}}{}^{\ldots\mu\ldots}{}_{\alpha k}h_{\lambda}] \\
& +[\bar{P}\bar{\psi}]\Big]
\end{aligned}
$$

Dirac field:

$$L = \sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}(\bar{\psi}\mathrm{i}\gamma_\mu D_\nu\psi - D_\nu\bar{\psi}\mathrm{i}\gamma_\mu\psi) - m\bar{\psi}\psi\right]$$

$\gamma^\mu = V^\mu_{\bar{\mu}}\gamma^{\bar{\mu}}$
$\gamma^{\bar{\mu}}$: standard Minkowski space gamma matrices

$$L^{\bar{\mu}\bar{\nu}} = \frac{1}{2}\sigma^{\bar{\mu}\bar{\nu}}\ , \qquad \sigma^{\bar{\mu}\bar{\nu}} = \frac{1}{2}[\gamma^{\bar{\mu}}, \gamma^{\bar{\nu}}]$$

$$\mathbb{Q}^{\mu\nu\lambda} = \frac{1}{8}\mathrm{i}\bar{\psi}(\gamma^\lambda\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\gamma^\lambda)\psi$$

$$\mathbb{Q}^{(\mu\nu)\lambda} = 0$$

$$I^\mu = -\sqrt{-g}\,\nabla_\nu(\mathbb{Q}^{\lambda\mu\nu}h_\lambda)$$