A relativistic conical function and its descendants

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Outline

1. Overview

- The conical function is a ${}_2F_1$-specialization that can be used to solve the Schrödinger equation for the (reduced) $N = 2$ case of the repulsive and attractive nonrelativistic integrable Calogero-Moser N-particle system. Its relativistic generalization serves the same purpose for the relativistic version of this integrable quantum system.
- To date, the only way to prove orthogonality and completeness of the associated relativistic eigenfunction transform involves scattering theory. We therefore begin by outlining how this works in the nonrelativistic case.
- After summarizing the Hilbert space aspects of the relativistic conical function, we sketch further features. This includes product formulas it satisfies, and how it gives rise to an $SL(2,\mathbb{Z})$-representation and a solution to quantum KZ equations. To conclude, we add some remarks on Cherednik's DAHA in the A_1 setting at issue.

2. Nonrelativistic 1D potential scattering

- We consider Schrödinger operators of the form ($\hbar \equiv 1$)

$$H_0 = -d^2/dx^2, \quad H = -d^2/dx^2 + V(x),$$

with $V(x)$ real-valued.

- Two 'position space' Hilbert spaces occur:

$$\mathcal{H}_s \equiv L^2((0,\infty), dx), \quad \mathcal{H}_d \equiv L^2((-\infty,\infty), dx).$$

- With suitable assumptions on $V(x)$, we recall the connection of the wave operators

$$W_\pm = \lim_{t \to \pm\infty} e^{itH} e^{-itH_0},$$

from time-dependent scattering theory with time-independent scattering theory in terms of (improper) eigenfunctions

$$H\Psi = p^2 \Psi, \quad p > 0,$$

with unitary asymptotics.

2A. Scattering on the half-line

- Assume $V(x)$ is smooth on $(0, \infty)$, vanishes quickly for $x \to \infty$, and satisfies

$$V(x) \to \infty, \quad x \to 0, \qquad V'(x) < 0, \quad x > 0.$$

- With Dirichlet b. c. at $x = 0$, the interacting and free evolutions $\exp(-itH)$ and $\exp(-itH_0)$ on $\mathcal{H}_s$ can be compared via the wave operators $W_\pm$. They are unitary, with the scattering encoded in the (position space) S-operator

$$S \equiv W_+^* W_-.$$

- This can be made more explicit by using the so-called incoming wave functions

$$H\Psi = p^2\Psi, \ p > 0, \quad \Psi(x,p) \sim u(p)e^{ixp} - e^{-ixp}, \quad x \to \infty,$$

with $u(p) =: \hat{S}_s(p)$ the unitary S-matrix ($|\hat{S}_s(p)| = 1$).

- The sine transform

$$(\mathcal{F}_0 f)(x) \equiv \sqrt{\frac{1}{2\pi}} \int_0^\infty dp \left(e^{ixp} - e^{-ixp}\right) f(p), \quad f \in C_0^\infty((0,\infty)),$$

diagonalizes H_0 on $\hat{\mathcal{H}}_s \equiv L^2((0,\infty), dp)$ ('momentum space'):

$$H_0 \mathcal{F}_0 = \mathcal{F}_0 p^2.$$

- Letting

$$(\mathcal{F} f)(x) \equiv \sqrt{\frac{1}{2\pi}} \int_0^\infty dp \, \Psi(x,p) f(p), \quad f \in C_0^\infty((0,\infty)),$$

we get more generally a unitary operator from $\hat{\mathcal{H}}_s$ to $\mathcal{H}_s$ such that

$$H\mathcal{F} = \mathcal{F} p^2.$$

- We also have

$$\mathcal{F} = W_- \mathcal{F}_0, \quad \mathcal{F}\hat{S}^* = W_+ \mathcal{F}_0, \quad (\hat{S} f)(p) \equiv \hat{S}_s(p) f(p),$$

with $\hat{S} = \mathcal{F}_0^* S \mathcal{F}_0$ the momentum space scattering operator.

2B. Scattering on the line

- Assume $V(x)$ is smooth, even, vanishes quickly for $|x| \to \infty$, and satisfies $V'(x) > 0$ for $x > 0$. Such V have finitely many bound states, i. e.,

$$H\Psi_\ell = E_\ell \Psi_\ell, \quad E_\ell < 0, \quad \Psi_\ell \in \mathcal{H}_d = L^2(\mathbb{R}, dx), \quad \ell = 0, \ldots, L-1.$$

- The wave operators $W_\pm$ exist and are isometric, with range equal to the orthogonal complement of the bound states. Thus, the position space S-operator $S = W_+^* W_-$ is unitary.
- A corresponding unitary S-matrix

$$\hat{S}_d(p) \equiv \begin{pmatrix} t(p) & r(p) \\ r(p) & t(p) \end{pmatrix}, \quad p > 0,$$

on the momentum space $\hat{\mathcal{H}}_d \equiv L^2((0,\infty), dp)^2$ arises as follows.

- Diagonalize H_0 and H via eigenfunction transforms

$$(\mathcal{F}_{(0)}f)(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty dp \begin{pmatrix} \Psi_{(0)}(x,p) \\ -\Psi_{(0)}(-x,p) \end{pmatrix} \cdot \begin{pmatrix} f_+(p) \\ f_-(p) \end{pmatrix},$$

with

$$H_{(0)}\Psi_{(0)} = p^2\Psi_{(0)}.$$

- For H_0 choose $\Psi_0(x,p) = \exp(ixp)$, so $\mathcal{F}_0$ amounts to the Fourier transform, with $\hat{f} \in L^2(\mathbb{R}, dp)$ yielding $(f_+, f_-) \in \hat{\mathcal{H}}_d$ via

$$f_+(p) \equiv \hat{f}(p), \quad f_-(p) \equiv -\hat{f}(-p), \quad p > 0.$$

- For H choose the incoming wave function $\Psi(x,p)$:

$$H\Psi = p^2\Psi, \ p > 0, \ \ \Psi(x,p) \sim \begin{cases} t(p)e^{ixp}, & x \to \infty, \\ e^{ixp} - r(p)e^{-ixp}, & x \to -\infty. \end{cases}$$

(So $\Psi(x,p)/t(p)$ is a Jost function.)

- Once more, we get $H_{(0)}\mathcal{F}_{(0)} = \mathcal{F}_{(0)}p^2$ and

$$\mathcal{F} = W_-\mathcal{F}_0, \mathcal{F}\hat{S}^* = W_+\mathcal{F}_0, (\hat{S}f)(p) \equiv \hat{S}_d(p) \begin{pmatrix} f_+(p) \\ f_-(p) \end{pmatrix},$$

so that the scattering is encoded in the momentum space scattering operator

$$\hat{S} = \mathcal{F}_0^* S \mathcal{F}_0.$$

- Hence H is diagonalized as multiplication by $(p^2, p^2) \oplus (E_0, \ldots, E_{L-1})$ on $\hat{\mathcal{H}}_d \oplus$ Span(bound states).

- N. B. In both cases, the eigenfunction transforms yield a concrete realization of the spectral theorem. Scattering theory can be avoided by using the so-called Weyl/Titchmarsh/Kodaira approaches.

3. The special potentials at issue

- We consider the two potentials on the half-line and the line leading to the conical function, namely,

$$V_s(x) \equiv g(g-1)/\sinh^2(x), \quad x \in (0,\infty), \;\; g > 1,$$

and

$$V_d(x) \equiv -g(g-1)/\cosh^2(x), \quad x \in \mathbb{R}, \;\; g > 1.$$

- Here, the suffix s stands for 'same', and d for 'different'. These potentials encode the interaction between two charged particles in their center-of-mass frame, with repulsion between same charges and attraction between different charges (as in electrodynamics).
- N. B. $V_d(x)$ arises from $V_s(x)$ by the analytic continuations $x \to x \pm i\pi/2$.

3A. The repulsive case

- The above incoming wave function $\Psi(x, p)$ involves the so-called conical function:

$$P^{1/2-g}_{ip-1/2}(\cosh x) \equiv \frac{(\sinh x)^{g-1/2}}{2^{g-1/2}\Gamma(g+1/2)}\psi_{\rm nr}(g; x, p),$$

$$\psi_{\rm nr}(g; x, p) \equiv {}_2F_1((g+ip)/2, (g-ip)/2, g+1/2; -\sinh^2(x)).$$

- These functions admit a variety of integral representations. Probably the simplest is

$$\psi_{\rm nr}(g; x, p) = \frac{2\Gamma(2g)}{2^g\Gamma(g+ip)\Gamma(g-ip)}\int_0^\infty dy\, \frac{\cos(yp)}{(\cosh y + \cosh x)^g},$$

which entails in particular

$$\psi_{\rm nr}(1; x, p) = \sin(xp)/p \sinh x.$$

- Setting

$$\Psi(x,p) \equiv -\frac{(2\sinh x)^g \Gamma(g)\Gamma(g-ip)}{\Gamma(2g)\Gamma(-ip)}\psi_{\rm nr}(g;x,p)$$

$$= -\frac{2(\sinh x)^g \Gamma(g)}{\Gamma(-ip)\Gamma(g+ip)}\int_0^\infty dy\, \frac{\cos(yp)}{(\cosh y + \cosh x)^g},$$

yields the announced incoming wave function:

$$\Psi(x,p) \sim u(p)e^{ixp} - e^{-ixp}, \quad x \to \infty,$$

where

$$u(p) = -\frac{\Gamma(ip)\Gamma(g-ip)}{\Gamma(-ip)\Gamma(g+ip)}.$$

- N. B. For $g = 1$ this gives the free solution

$$\Psi(x,p) = e^{ixp} - e^{-ixp}.$$

3B. The attractive case

- For $g \in (L, L+1]$ there are L bound states

$$\Psi_\ell(x) = (\cosh x)^{1-g} P_\ell(i \sinh x),$$

$$H\Psi_\ell = E_\ell \Psi_\ell, \quad E_\ell = -(g-\ell-1)^2, \;\; \ell = 0, \ldots, L-1,$$

with $P_\ell(t)$ Gegenbauer polynomials of degree ℓ, satisfying

$$P_\ell(-t) = (-)^\ell P_\ell(t).$$

- The solution space to $H\Psi = p^2\Psi$, $p > 0$, is spanned by the two functions

$$(\cosh x)^g \psi_{\rm nr}(g; x \pm i\pi/2, p).$$

Therefore the desired incoming wave function $\Psi(x, p)$ is characterized by two p-dependent coefficients.

- Specifically, it reads

$$\Psi(x,p) = \frac{(2\cosh x)^g \Gamma(g)\Gamma(g-ip)}{2\Gamma(2g)\Gamma(-ip)\sinh(i\pi g - \pi p)}$$

$$\times \sum_{\delta=+,-} \delta \exp(\delta(i\pi g - \pi p)/2)\psi_{\mathrm{nr}}(g; x + \delta i\pi/2, p)$$

$$\sim \begin{cases} t(p)e^{ixp}, & x \to \infty, \\ e^{ixp} - r(p)e^{-ixp}, & x \to -\infty, \end{cases}$$

with

$$t(p) = \frac{\sinh(\pi p)}{\sinh(i\pi g - \pi p)} u(p), \quad r(p) = \frac{\sinh(i\pi g)}{\sinh(i\pi g - \pi p)} u(p).$$

- N. B. For $g = 1, 2, 3, \ldots$, we get $r(p) = 0$. Moreover, $g = 1$ yields the free solution

$$\Psi(x,p) = e^{ixp}.$$

4. The relativistic generalization

- The relativistic Calogero-Moser system is encoded in N commuting Hamiltonians that are AΔOs (analytic difference operators):

$$H_k(x) = \sum_{|I|=k} \prod_{\substack{m\in I\\ n\notin I}} f_-(x_m - x_n) e^{-i\hbar\beta \sum_{m\in I} \partial_{x_m}} \prod_{\substack{m\in I\\ n\notin I}} f_+(x_m - x_n),$$

where $k = 1, \ldots, N, \ \ \beta > 0$, and

$$f_\pm(x)^2 = \sinh(\mu(x \pm i\beta g)/2))/\sinh(\mu x/2).$$

- Physical picture: $\beta = 1/mc$ and c =light speed; $H = mc^2[H_1(x) + H_1(-x)]$, $P = mc[H_1(x) - H_1(-x)]$, and $B = -m\sum_{j=1}^{N} x_j$, are space-time translation and boost generators, representing the Lie algebra of the Poincaré group in 2D:

$$[H, P] = 0, \ \ [H, B] = i\hbar P, \ \ [P, B] = i\hbar c^{-2} H.$$

4A. The (reduced) $N = 2$ repulsive case

- To date no general Hilbert space theory for AΔOs exists. Worse yet, the solutions to a Schrödinger equation of the form

$$f(x)\Psi(x + is, p) + g(x)\Psi(x - is, p) = 2\cosh(sp)\Psi(x, p),$$

  with shift parameter $s > 0$ form an infinite-dimensional vector space whenever one nonzero solution $\Psi(x, p)$ exists.
- Example: The free case $f(x) = g(x) = 1$. Just multiply the obvious solution $\exp(ixp)$ by any function $m(x, p)$ that has is-periodicity in x to get another solution.
- Certain special AΔOs, however, have been promoted to self-adjoint Hilbert space operators. This hinges on the existence of special solutions to the Schrödinger equation that give rise to a unitary eigenfunction transform.

- For the reduced $N = 2$ case at hand, this transform involves the relativistic conical function. This conical function generalization has many distinct integral representations. The integrands are built from the hyperbolic gamma function $G(a_+, a_-; z)$, which is a generalization of the (rational) gamma function $\Gamma(z)$.
- In the present setting, $a_\pm$ can be viewed as length scales:

$$a_+ \equiv 2\pi/\mu, \quad \text{(imaginary period/interaction length)},$$

$$a_- \equiv \hbar/mc, \quad \text{(shift step size/Compton wave length)}.$$

- From now on, we use the notation

$$c_\delta(z) \equiv \cosh(\pi z/a_\delta), \ s_\delta(z) \equiv \sinh(\pi z/a_\delta), \ e_\delta(z) \equiv e^{\pi z/a_\delta},$$

where $\delta = +, -$; also, we define the average

$$a \equiv (a_+ + a_-)/2.$$

- The hyperbolic gamma function $G(z)$ can be defined as the meromorphic solution to one of the first order AΔEs

$$\frac{G(z+ia_\delta/2)}{G(z-ia_\delta/2)}=2c_{-\delta}(z),\quad \delta=+,-,\quad a_+,a_->0,$$

  which is uniquely determined by requiring $G(0)=1$ and 'minimality'; the second AΔE is then satisfied as well.
- In the strip $|\mathrm{Im}\, z|<a$ it has the integral representation

$$G(z)=\exp\Big(i\int_0^\infty\frac{dy}{y}\Big(\frac{\sin 2yz}{2\sinh(a_+y)\sinh(a_-y)}-\frac{z}{a_+a_-y}\Big)\Big).$$

  This entails absence of zeros and poles in this strip and the properties

$$G(a_-,a_+;z)=G(a_+,a_-;z),\quad (\text{modular invariance}),$$

$$G(-z)=1/G(z),\quad (\text{reflection equation}),$$

$$\overline{G(z)}=G(-\bar{z}).$$

- The simplest and most revealing representation of the relativistic conical function is given by

$$\mathcal{R}(a_+, a_-, b; x, y) = \sqrt{\frac{1}{a_+ a_-} \frac{G(2ib - ia)}{G(ib - ia)^2}}$$

$$\times \int_{\mathbb{R}} dz \prod_{\delta=+,-} \frac{G(z + \delta(x - y)/2 - ib/2)}{G(z + \delta(x + y)/2 + ib/2)}.$$

Here, b and y are the coupling constant and spectral parameter, related to the previous parameters by

$$b = \beta g (= g/mc), \quad y = \beta p/\mu.$$

- From this one reads off evenness in x and y and the properties

$$\mathcal{R}(a_-, a_+, b; x, y) = \mathcal{R}(a_+, a_-, b; x, y), \quad (\text{modular invariance}),$$

$$\mathcal{R}(a_+, a_-, b; y, x) = \mathcal{R}(a_+, a_-, b; x, y), \quad (\text{self} - \text{duality}).$$

- The $\mathcal{R}$-function is meromorphic for $b, x, y \in \mathbb{C}$ and $\mathrm{Re}\, a_+, \mathrm{Re}\, a_- > 0$. It satisfies the four AΔEs

$$A_\delta(x)\mathcal{R}(x,y) = 2c_\delta(y)\mathcal{R}(x,y), \quad A_\delta(y)\mathcal{R}(x,y) = 2c_\delta(x)\mathcal{R}(x,y),$$

$$A_\delta(z) \equiv \frac{s_\delta(z+ib)}{s_\delta(z)} \exp(ia_{-\delta} d/dz) + (z \to -z),$$

where $\delta = +, -$.
- The AΔO $A_+(x)$ is related to the above (reduced) $N = 2$ Hamiltonian H by a similarity transformation involving the generalized Harish-Chandra c-function

$$c(z) \equiv G(z + ia - ib)/G(z + ia).$$

- Introducing the weight and scattering functions

$$w(z) \equiv 1/c(z)c(-z), \quad u(z) \equiv -c(z)/c(-z),$$

(with $w(z)$ having a double zero for $z = 0$), this relation is given by

$$H = C^{st} H_+(x), \quad H_\pm(z) \equiv w(z)^{1/2} A_\pm(z) w(z)^{-1/2}.$$

- The function

$$\Psi(x,y) \equiv -\frac{G(ib-ia)}{G(2ib-ia)}\frac{w(x)^{1/2}}{c(-y)}\mathcal{R}(x,y),$$

satisfies $H_\pm(x)\Psi(x,y) = 2c_\pm(y)\Psi(x,y)$ and

$$\Psi(x,y) \sim u(y)\exp(i\pi xy/a_+a_-) - \exp(-i\pi xy/a_+a_-), \quad x \to \infty.$$

- Setting

$$H_{0,\pm}(x) \equiv \exp(ia_\mp d/dx) + \exp(-ia_\mp d/dx),$$

$$\Psi_0(x,y) \equiv \exp(i\pi xy/a_+a_-) - \exp(-i\pi xy/a_+a_-),$$

one clearly gets $H_{0,\pm}(x)\Psi_0(x,y) = 2c_\pm(y)\Psi_0(x,y)$.
- The sine transform $\mathcal{F}_0$ with kernel $(2a_+a_-)^{-1/2}\Psi_0(x,y)$ can now be used to reinterpret the AΔOs $H_{0,\pm}(x)$ as self-adjoint operators on $\mathcal{H}_s = L^2((0,\infty),dx)$, namely as pullbacks of the self-adjoint operators of multiplication by $2c_\pm(y)$ on $\hat{\mathcal{H}}_s = L^2((0,\infty),dy)$ under the unitary $\mathcal{F}_0$.

- Provided $b \in [0, 2a]$, the transform $\mathcal{F}$ with kernel $(2a_+ a_-)^{-1/2}\Psi(x, y)$ yields a unitary operator $\hat{\mathcal{H}}_s \to \mathcal{H}_s$. (It equals $\mathcal{F}_0$ for $b = a_\pm$.) The AΔOs $H_\pm(x)$ can then be viewed as commuting self-adjoint operators on $\mathcal{H}_s$, defined by $\mathcal{F}2c_\pm(\cdot)\mathcal{F}^*$.
- These transforms are related to the wave operators

$$W_\pm = \lim_{t\to\pm\infty} \exp(itH_\delta)\exp(-itH_{0,\delta}), \quad \delta = +, -,$$

in the same way as in the nonrelativistic setting.
- In particular, the scattering operator on $\hat{\mathcal{H}}_s$ is given by

$$(\hat{S}f)(y) = \hat{S}_s(y)f(y), \quad \hat{S}_s(y) \equiv u(y),$$

with

$$u(y) = -\frac{G(y + ia - ib)G(y - ia + ib)}{G(y + ia)G(y - ia)}.$$

4B. The (reduced) $N = 2$ attractive case

- Reminder:

$$s_+(x) = \sinh(\pi x/a_+), \quad a_+ = 2\pi/\mu, \quad a_- = \hbar/mc, \quad y = p/mc\mu.$$

- The repulsive (same charge) and attractive (different charge) AΔOs are given by

$$A_s(x) \equiv A_+(x) = \frac{s_+(x+ib)}{s_+(x)} \exp(ia_- d/dx) + (x \to -x),$$

$$A_d(x) \equiv A_+(x - ia_+/2) = \frac{c_+(x+ib)}{c_+(x)} \exp(ia_- d/dx) + (x \to -x).$$

- Setting

$$\tilde{c}(x) \equiv c(x - ia_+/2), \quad \tilde{w}(x) \equiv 1/\tilde{c}(x)\tilde{c}(-x) > 0, \ \ \forall x \in \mathbb{R},$$

the corresponding Hamiltonian is

$$H_d(x) \equiv \tilde{w}(x)^{1/2} A_d(x) \tilde{w}(x)^{-1/2}.$$

For $b = a_-$ it equals $e^{ia_- d/dx} + e^{-ia_- d/dx}$.

- N. B. The x-shift $\mathcal{R}(x - ia_+/2, y)$ entails that modular invariance and self-duality break down. As a result, $A_d(x)$ has no natural 'modular partner', and we might as well trade the spectral variable y (a position) for p (a momentum). For brevity, we stick to y.
- Clearly, we get two distinct eigenfunctions

$$A_d(x)\mathcal{R}(x \pm ia_+/2, y) = 2c_+(y)\mathcal{R}(x \pm ia_+/2, y),$$

which entails

$$H_d(x)\tilde{w}(x)^{1/2}\mathcal{R}(x \pm ia_+/2, y)$$
$$= 2c_+(y)\tilde{w}(x)^{1/2}\mathcal{R}(x \pm ia_+/2, y).$$

- Snag. These $H_d(x)$-eigenfunctions remain eigenfunctions when multiplied by any function $m(x, y)$ that is ia_--periodic in x. There are no general results ensuring that a particular choice yields a function $\Psi(x, y)$ that can serve as the kernel of a unitary eigenfunction transform.

- The linear combination

$$\Psi(x,y) \equiv \frac{G(ib-ia)}{G(2ib-ia)} \frac{\tilde{w}(x)^{1/2}}{2s_-(ib-y)c(-y)}$$

$$\times \sum_{\delta=+,-} \delta e_-(\delta(ib-y)/2)\mathcal{R}(x+\delta ia_+/2, y),$$

has coefficients ensuring unitary asymptotics:

$$\Psi(x,y) \sim \begin{cases} t(y)e^{i\pi xy/a_+a_-}, & \mathrm{Re}\, x \to \infty, \\ e^{i\pi xy/a_+a_-} - r(y)e^{-i\pi xy/a_+a_-}, & \mathrm{Re}\, x \to -\infty, \end{cases}$$

with

$$t(y) \equiv \frac{s_-(y)}{s_-(ib-y)}u(y), \quad r(y) \equiv \frac{s_-(ib)}{s_-(ib-y)}u(y).$$

- N. B. The triple u, t, r satisfies the Yang-Baxter equations; note also $r = 0$ for $b = (L+1)a_-$, $L = 0, 1, 2, \ldots$.

- In joint work with S. Haworth we have shown that the transform

$$(\mathcal{F}f)(x) = \frac{1}{\sqrt{2a_+a_-}} \int_0^\infty dy \begin{pmatrix} \Psi(x,y) \\ -\Psi(-x,y) \end{pmatrix} \cdot \begin{pmatrix} f_+(y) \\ f_-(y) \end{pmatrix},$$

yields a unitary operator

$$\mathcal{F} \,:\, \hat{\mathcal{H}}_d \equiv L^2((0,\infty), dy)^2 \to \mathcal{H}_d \equiv L^2(\mathbb{R}, dx),$$

provided $b \in [0, a_-]$. Also, $\Psi(x,y)$ equals $e^{i\pi xy/a_+a_-}$ for $b = a_-$, so then $\mathcal{F}$ amounts to the Fourier transform $\mathcal{F}_0$.

- For $b \in (a_-, a_- + a_+/2)$ the transform is isometric. Its range is the orthogonal complement of $L \geq 1$ bound states

$$\Psi_\ell(x) = \frac{c_+(x)}{\tilde{w}(x)^{1/2}} Q_\ell(is_+(x)), \quad \ell = 0, \ldots, L-1,$$

with $Q_\ell(t)$ q-Gegenbauer polynomials of degree ℓ and parity $(-)^\ell$.

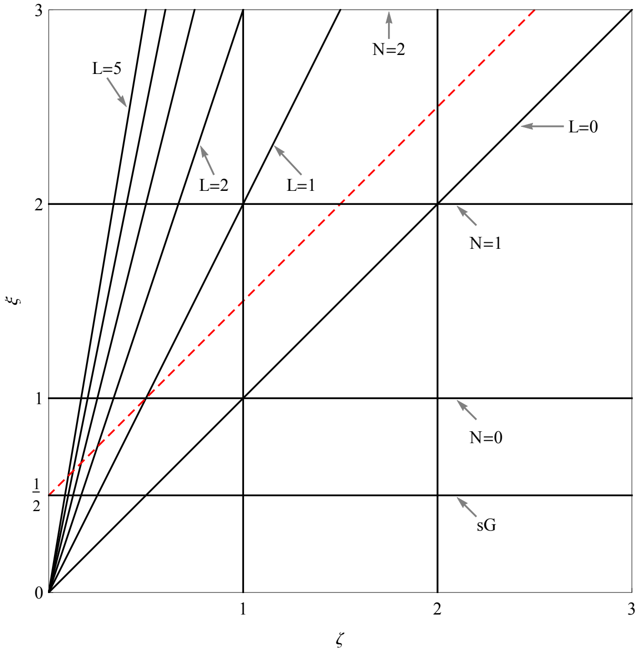
- The transforms $\mathcal{F}_{(0)}$ are related to the wave operators $W_{\pm}$ as before, and serve to associate a self-adjoint operator on $\mathcal{H}_d$ to the AΔO $H_d(x)$, namely the pullback of multiplication by $(2c_+(y), 2c_+(y))$ on $\hat{\mathcal{H}}_d$.
- For a fixed $b \in [0, a_- + a_+/2)$, the bound state number L is the smallest integer such that $b \leq (L+1)a_-$. For $b > a_-$ we have $H_d\Psi_\ell = E_\ell\Psi_\ell$, with

$$E_\ell = 2c_+(i(b - (\ell+1)a_-)) \in (0, 2), \quad \ell = 0, \ldots, L-1.$$

- Setting

$$\xi \equiv b/a_+, \quad \zeta \equiv a_-/a_+,$$

the following plot can be viewed as a phase diagram. The red line denotes the transition to the 'unphysical' regime (breakdown of isometry and self-adjointness). On the lines $\xi = (L+1)\zeta$, $L = 0, 1, \ldots$, the reflection vanishes. Also, sG stands for the sine-Gordon line $\xi = 1/2$. The nonrelativistic limit arises by setting $\xi = \lambda\zeta$, $\lambda = g/\hbar$ fixed, and letting $\zeta \to 0$.


3
2
1
$\frac{1}{2}$
0
1
2
3
ξ
ζ
L=5
L=2
L=1
L=0
N=2
N=1
N=0
sG

5. Further developments

- In this section we sketch issues involving cousins of $\mathcal{R}(x, y)$ defined for $b \in (0, 2a)$ and $x, y > 0$, namely

$$J(x, y) \equiv \sqrt{a_+ a_-}\mathcal{R}(x, y)G(ia - 2ib) \prod_{\delta=+,-} G(\delta y - ia + ib),$$

  and the self-dual real-valued function

$$\mathrm{F}(x, y) \equiv G(ia - 2ib)G(ib - ia)w(x)^{1/2}\mathcal{R}(x, y)w(y)^{1/2},$$

  which is related to the incoming wave function $\Psi(x, y)$ by

$$\mathrm{F}(x, y) = u(y)^{-1/2}\Psi(x, y).$$

- Identifying $L^2((0, \infty), dx)$ and $L^2((0, \infty), dy)$, we obtain a unitary and self-adjoint involution $\mathcal{I}$ with integral kernel $(2a_+ a_-)^{-1/2}\mathrm{F}(x, y)$.

5A. Product formulas

- In joint work with M. Hallnäs we have shown that the J-function satisfies the product formulas

$$J(b;x,v)J(b;y,v) = \frac{1}{2}\int_0^\infty dz\, w(b;z)J(b;z,v)$$

$$\prod_{\delta_1,\delta_2,\delta_3=+,-} G((\delta_1 x+\delta_2 y+\delta_3 z - ib)/2),$$

$$J(b;x,t)J(b;x,u) = \frac{1}{2}G(ia-ib)^2\int_0^\infty dv\, w(2a-b;v)J(b;x,v)$$

$$\times \prod_{\delta_1,\delta_2,\delta_3=+,-} G((\delta_1 t+\delta_2 u+\delta_3 v + ib)/2 - ia).$$

- These formulas have various spin-offs, including crucial applications to the $N=3$ joint eigenfunctions and limits yielding novel product formulas for the conical function.

5B. An $SL(2,\mathbb{Z})$ representation

- The reduction approach to the various regimes of the classical versions of the Calogero-Moser N-particle systems led L. Feher and C. Klimcik to an $SL(2,\mathbb{Z})$ representation in a self-dual regime that is closely related to the relativistic hyperbolic regime whose $N = 2$ quantum version is at issue here.
- We have shown by a direct method that this representation also holds true for the classical $N = 2$ hyperbolic relativistic case. Moreover, up to some unresolved domain issues, this representation persists at the quantum level.
- Specifically, the $SL(2,\mathbb{Z})$ generator $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is represented by the unitary involution $\mathcal{I}$ and the generator $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ by the Gaussian unitary $G \equiv \exp(i\pi x^2/2a_+a_-)$. (Crux: the operators $\mathcal{I}G^*\mathcal{I}$ and $G\mathcal{I}G$ are equal up to an unknown phase.)

5C. Cherednik's A_1 DAHA: First steps

- In the present 'modular' setting there are two choices of DAHA, labeled by $\delta = +, -$. Letting

$$X \equiv e_\delta(x), \quad D \equiv \exp(ia_{-\delta}\partial_x),$$

and $(sf)(-x) \equiv f(-x)$, we can take as Demazure-Lusztig operator

$$T \equiv e_\delta(ib) + \frac{s_\delta(x + ib)}{s_\delta(x)}(s - 1),$$

and as Dunkl-Cherednik operator $Y \equiv sDT$.

- This entails

$$(T - e_\delta(ib))(T + e_\delta(-ib)) = \mathbf{1}, \quad TY^{-1}T = Y,$$

$$Y^{-1}X^{-1}YXT^2 = e_\delta(-ia_{-\delta}),$$

as required.

- Moreover, on symmetric (even) functions the operator $Y + Y^{-1}$ acts as $A_\delta(x)$, so it has $\mathcal{R}(x, y)$ as eigenfunction with eigenvalue $2c_\delta(y)$.
- There is no obvious way to similarity transform $Y + Y^{-1}$ to a (formally) normal operator on $L^2(\mathbb{R}, dx)$. We have also been unable to find a 'non-symmetric' eigenfunction of $Y + Y^{-1}$ whose symmetric part equals $\mathcal{R}(x, y)$.
- On the other hand, following van Meer/Stokman (IMRN, 2010), we can use $\mathcal{R}(x, y)$ to construct solutions to a modular version of the bispectral quantum Knizhnik/Zamolodchikov equations, involving a matrix

$$M_\delta(x) \equiv \frac{1}{s_\delta(x - ib)} \begin{pmatrix} -s_\delta(ib)e_\delta(x) & s_\delta(x) \\ s_\delta(x) & -s_\delta(ib)e_\delta(-x) \end{pmatrix},$$

which satisfies $M_\delta(x)M_\delta(-x) = \mathbf{1}_2$.

- These bispectral quantum KZ equations are given by the system

$$\begin{pmatrix} 0 & e_\delta(-x) \\ e_\delta(x) & 0 \end{pmatrix} M_\delta(y)F_\delta(x,y) = F_\delta(x, y - ia_{-\delta}),$$

and its $x \leftrightarrow y$ counterpart.

- The solution to these systems is given by the self-dual function $F_\delta = (F_{1,\delta}, F_{2,\delta})$, with

$$F_{1,\delta}(x,y) \equiv \frac{1}{2s_\delta(y+ib)}[\mathcal{R}(x+ia_{-\delta}, y) - e_\delta(-y-ib)\mathcal{R}(x,y)],$$

$$F_{2,\delta}(x,y) \equiv e_\delta(-ib)[\mathcal{R}(x,y) - F_{1,\delta}(x,y)].$$

- This self-duality feature is encoded in the novel identity

$$s_\delta(x+ib)\mathcal{R}(x+ia_{-\delta}, y) - s_\delta(y+ib)\mathcal{R}(x, y+ia_{-\delta})$$

$$= s_\delta(x-y)\mathcal{R}(x,y).$$

6. Some recent references

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