Measurement-induced nonlinear transformations and their possible application for quantum informational tasks

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Introduction

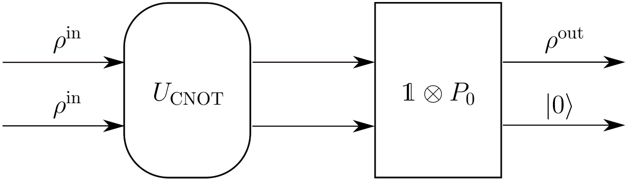
- How do nonlinear quantum transformations emerge?
- Focus on nonlinear evolution, but not on chaotic regime
- Not quantum chaos (!)
- Present a physical setup to realize a nonlinear scheme
  - could be used for quantum state discrimination
- Imagine another scenario where nonlinearity could be used
  - to decide whether the output of a quantum operation is close enough to the anticipated state

How does nonlinearity emerge?

- Quantum Mechanics: linear transformations
- BUT post-selection conditioned on measurement results ⇒ initial state can be nonlinearly transformed

Original proposal

- at least two identical independent copies of the same state ρ^{in}
- protocol:



$$\rho^{\text{in}} = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} \longrightarrow \rho^{\text{out}} = \begin{pmatrix} \rho_{11}^2 & \rho_{12}^2 \\ \rho_{21}^2 & \rho_{22}^2 \end{pmatrix}$$

H. Bechmann-Pasquinucci, B. Huttner and N. Gisin, Phys. Lett. A **242**, 198 (1998).

Nonlinear quantum transformations

- Two identical independent qubits (A and B) in the same pure state $|\psi_0\rangle$

$$|\psi_0\rangle = \frac{|0\rangle + z\,|1\rangle}{\sqrt{1+|z|^2}} \quad (z \in \mathbb{C})$$

- The state of the composite system

$$|\Psi_0\rangle = |\psi_0\rangle_A \otimes |\psi_0\rangle_B$$

- Apply an entangling two-qubit operation then measure the state of qubit B
  - qubit A is kept only if the measurement on B resulted 0
  - the state of A after the postselection reads

$$|\psi_1\rangle_A \sim |0\rangle_A + f(z)\,|1\rangle_A \qquad \text{(general case)}$$

  where $f(z)$ is a complex quadratic rational function of z

$$f(z) = \frac{a_0 z^2 + a_1 z + a_2}{b_0 z^2 + b_1 z + b_2}$$ if a_0 and b_0 are not both zero then nonlinear

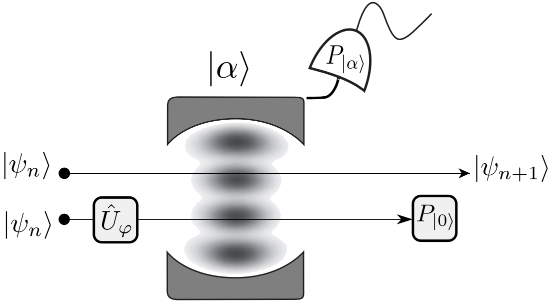
Iteration of the nonlinear map

- Ensemble of qubits $\{|\psi_0\rangle\}$
- Take pairs and apply the protocol
- Form new pairs from the postselected ones
- After the nth step

$$|\psi_0\rangle = \frac{|0\rangle + z\,|1\rangle}{\sqrt{1+|z|^2}} \longrightarrow |\psi_n\rangle = \frac{|0\rangle + f^{(n)}(z)\,|1\rangle}{\sqrt{1+\left|f^{(n)}(z)\right|^2}}$$

- Nonlinearity $\Rightarrow$ $|\psi_n\rangle$ can be very sensitive to the initial conditions
- Two ensembles $\{|\psi_I\rangle\}$ and $\{|\psi_{II}\rangle\}$
  - initially $|\psi_I\rangle$ and $|\psi_{II}\rangle$ may have a large overlap
  - but they may end up in orthogonal states (depending on $f(z)$)

A physical scheme for realization



1. 2 two-level atoms in the same state $|\psi_n\rangle$
2. $\hat{U}_\varphi$ is applied to atom B
3. interaction with an optical resonator that is in a coherent state $|\alpha\rangle$
4. projection of the field onto the initial coherent state
5. atom B is projected onto its ground state
6. atom A is left in the state $|\psi_{n+1}\rangle = \mathcal{N}\left[|0\rangle + f^{(n+1)}(z)\,|1\rangle\right]$

J. M. Torres, J. Z. Bernád, G. Alber, O. Kálmán, and T. Kiss, Phys. Rev. A, **95**, 023828 (2017)

Description of the physical setup

Resonant two-atom Tavis-Cummings model

- 2 (two-level) atoms + a single mode of the radiation field

$$\hat{H}_I = \hbar g \sum_{i=A,B} \left(\hat{\sigma}_i^+ \hat{a} + \hat{\sigma}_i^- \hat{a}^\dagger \right) \qquad \begin{aligned} \hat{\sigma}_i^+ &= |1\rangle\langle 0|_i \\ \hat{\sigma}_i^- &= |0\rangle\langle 1|_i \end{aligned}$$

  - reference Hamiltonian

$$\hat{H}_0 = \hbar\omega \left(\hat{a}^\dagger \hat{a} + |1\rangle\langle 1|_A + |1\rangle\langle 1|_B \right)$$

  - $\hat{H}_I$ commutes with $\hat{H}_0$

    $\Downarrow$

    the time-dependent state vector for a given initial pure state $|\Psi_0\rangle$

$$|\Psi_t\rangle = e^{-i\frac{\hat{H}_I t}{\hbar}} |\Psi_0\rangle$$

Time evolution I

- initial condition: normalized product state of the 2 atoms and the field

$$\left|\Psi_0\right\rangle = \left|\Psi_0^{\text{at}}\right\rangle |\alpha\rangle$$

$$\left|\Psi_0^{\text{at}}\right\rangle = c_0 |0,0\rangle + c_- \left|\Psi^-\right\rangle + c_+ \left|\Psi^+\right\rangle + c_1 |1,1\rangle$$

$$\left|\Psi^\pm\right\rangle = \frac{1}{\sqrt{2}} (|0,1\rangle \pm |1,0\rangle)$$

- the single mode of the radiation field is in a coherent state $|\alpha\rangle$

$$|\alpha\rangle = \sum_{n=0}^{\infty} e^{-\frac{|\alpha|^2}{2}} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \qquad \alpha = \sqrt{\bar{n}}\, e^{i\phi}$$

  - $\bar{n}$: mean photon number

Time evolution II

- exact solution of the time-dependent state vector

$$|\Psi_t\rangle = |0,0\rangle \left|\chi_t^{-1}\right\rangle + \left|\Psi^+\right\rangle \left|\chi_t^0\right\rangle + |1,1\rangle \left|\chi_t^1\right\rangle + c_- \left|\Psi^-\right\rangle |\alpha\rangle$$

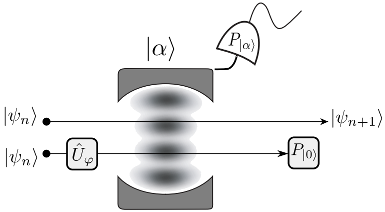
- for $\bar{n} \gg 1$ the photonic states can be simplified to

$$\left|\chi_t^k\right\rangle \approx \frac{e^{ik\phi}}{\sqrt{1+|k|}} \left(\eta_- \left|F_{k,t}^-\right\rangle + (-1)^k \eta_+ \left|F_{k,t}^+\right\rangle - k d_\phi^- |\alpha\rangle\right)$$

$$\left|F_{k,t}^\pm\right\rangle = e^{\pm i2gt\frac{1+k(\bar{n}+1)}{\sqrt{4\bar{n}+1}}} \left|\alpha e^{\frac{\pm i2gt}{\sqrt{4\bar{n}+1}}}\right\rangle \qquad k \in \{-1,0,1\}$$

$$\eta_\pm = \frac{1}{2}\left(c_+ \mp d_\phi^+\right) \qquad d_\phi^\pm = \frac{e^{i\phi}c_0 \pm e^{-i\phi}c_1}{\sqrt{2}}$$

Measurements & postselection



$$|\psi_0\rangle = \frac{|0\rangle + ze^{i\phi}|1\rangle}{\sqrt{1+|z|^2}}$$

$$\hat{U}_\varphi^B = \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & -e^{-i\varphi} \end{pmatrix}$$

- projecting the field onto the initial coherent state $|\alpha\rangle$:

$$\left|\Psi_1^{\text{at}}\right\rangle = \frac{\sqrt{2}ze^{i\phi}\cos\varphi}{(1+|z|^2)Q_1}|\Psi^-\rangle - \frac{e^{-i\varphi}+z^2e^{i\varphi}}{2(1+|z|^2)Q_1}\left(|0,0\rangle - e^{i2\phi}|1,1\rangle\right)$$

- projecting the state of atom B to $|0\rangle$

$$\left|\Psi_1^A\right\rangle = -\frac{ze^{i\phi}\cos\varphi}{(1+|z|^2)Q_1Q_2}|1\rangle - \frac{e^{-i\varphi}+z^2e^{i\varphi}}{2(1+|z|^2)Q_1Q_2}|0\rangle$$

- overall success probability

$$P_s = Q_2^2Q_1^2 = Q_1^2/2 \geq \frac{\cos^2\varphi}{4}$$

The resulting nonlinear transformation

- the final state of atom A (up to normalization) is given by

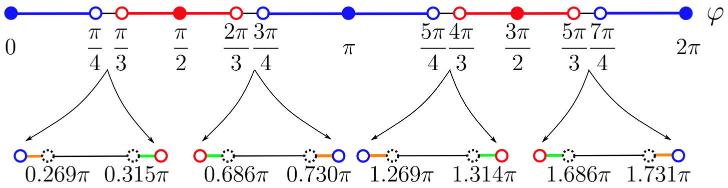
$$|0\rangle + \frac{2z\cos\varphi}{e^{-i\varphi} + z^2 e^{i\varphi}} e^{i\phi} |1\rangle$$

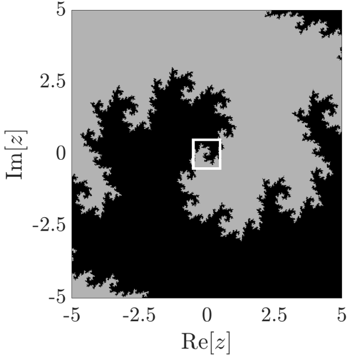
nonlinear quantum map

$$f_\varphi(z) = \frac{2z\cos\varphi}{e^{-i\varphi} + z^2 e^{i\varphi}}$$ complex quadratic rational function

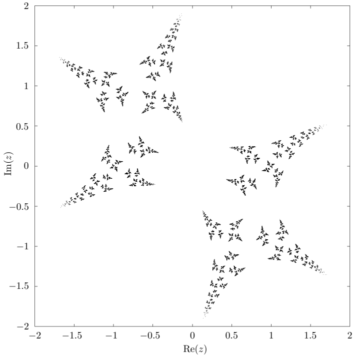
- fixed n-cycles: $f_\varphi^{(n)}(z) = z$
- stability of the fixed n-cycles: multiplier $\lambda = \left(f_\varphi^{(n)}\right)'(z_j) = f'_\varphi(z_1) f'_\varphi(z_2) ... f'_\varphi(z_n)$
  - $|\lambda| > 1$ repelling
  - $|\lambda| = 1$ neutral
  - $|\lambda| < 1$ attractive
  - $|\lambda| = 0$ superattractive
- the critical points from $f'_\varphi(z) = 0$
  - we can follow their orbits numerically
  - stable cycles can be found (here at most 2)

Different parameter regimes





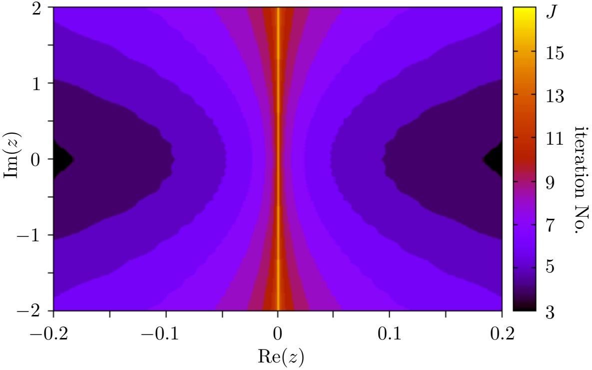
$\varphi = 0.95\pi/4$



$\varphi = 1.666\pi$

An application for state discrimination

- $\varphi = 0$
- $U_\varphi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (Z gate)
- the nonlinear transformation is $f_{\varphi=0} = \dfrac{2z}{1+z^2}$
  - two fixed one-cycles: 1 and -1



$$|\Psi_0\rangle_I = \frac{|0\rangle + 0.2\,|1\rangle}{\sqrt{1+(0.2)^2}}$$

$$|\Psi_0\rangle_{II} = \frac{|0\rangle - 0.2\,|1\rangle}{\sqrt{1+(0.2)^2}}$$

$${}_I\langle\Psi_0\,|\,\Psi_0\rangle_{II} \approx 0.92$$

$${}_I\langle\Psi_3\,|\,\Psi_3\rangle_{II} \approx 0.08$$

From highly overlapping to almost orthogonal in only 3 steps

About quadratic rational functions

- the multipliers μ_i of the fixed points determine a conjugacy class of f
  - i.e. the μ_i's are left unchanged by the transformation

$$f' = g \circ f \circ g^{-1}$$

  - $g(z) = \dfrac{az+b}{cz+d} \quad (a,b,c,d \in \mathbb{C},\ ad-bc \neq 0)$ is a Möbius transformation
- any member of the conjugacy class of f can be found from

$$f_N(z) = \frac{z\,(z+\mu_1)}{\mu_2 z + 1}, \quad \mu_1\mu_2 \neq 1 \qquad \text{fixed-point normal form}$$

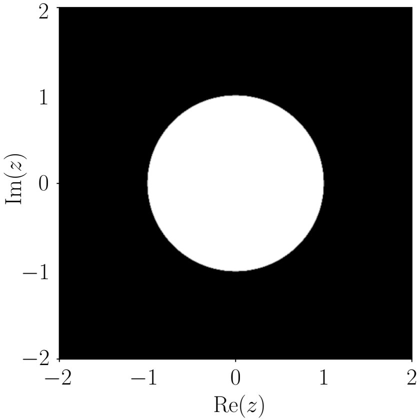
  - fixed points: $z_1 = 0$, $z_2 = \infty$, and $z_3 = \frac{1-\mu_1}{1-\mu_2}$, $\quad \left(\mu_3 = \frac{2-\mu_1-\mu_2}{1-\mu_1\mu_2}\right)$
  - superattractive for both fixed points if $\mu_1 = 0$ and $\mu_2 = 0$

$$f_0(z) = z^2 \qquad \text{basic superattractive map}$$

    - orthogonalizing because $\left|\psi_{z_1}\right\rangle = |0\rangle$ and $\left|\psi_{z_2}\right\rangle = |1\rangle$

J. Milnor, *Dynamics in One Complex Variable*, (Princeton University Press, 2006),
J. Milnor and Tan Lei, Experiment. Math. **2**, 37 (1993)

Properties of the basic map $f(z) = z^2$



After iteration:

- if $|z| < 1$ states converge to $|0\rangle$
- if $|z| > 1$ states converge to $|1\rangle$
- Julia set: $|z| = 1$ unit circle
  - contains $z_3 = 1$
  - closure of the set of all repelling fixed cycles
  - connected set

Orthogonalizing superattractive nonlinear maps

What Möbius transformations take $f_0(z) = z^2$ into another orthogonalizing superattractive map?

- multipliers are not changed
- need to keep $z_2 = -\frac{1}{z_1^*}$ (orthogonalizing property)

The effect of conjugating f_0 by $g(z) = \frac{az+b}{cz+d} \quad (ad - bc \neq 0)$

$$
\begin{aligned}
z_{0,1} &= 0 \xrightarrow{g} z_1 = \frac{b}{d} \\
z_{0,2} &= \infty \xrightarrow{g} z_2 = \frac{a}{c} = -\frac{1}{z_1^*} \\
z_{0,3} &= 1 \xrightarrow{g} z_3 = \frac{a+b}{c+d}.
\end{aligned}
$$

$\Rightarrow$ Möbius has to be of the form: $g(z) = \frac{az + z_1 d}{-az_1^* z + d} \quad (ad \neq 0)$

Quantum state matching I

- Julia set of $f_0(z) = z^2$: unit circle $\mathcal{J}_{f_0} = \left\{e^{i\varphi}, \varphi \in [0, 2\pi)\right\}$
- Julia set of $f = g \circ f_0 \circ g^{-1}$ can be given as $g\left(\left\{\mathcal{J}_{f_0}\right\}\right)$
  - g maps a circle into a circle or a line

    $\Rightarrow$ the Julia set of f is a circle or a line

Question: can we determine f by

- defining its z_1
- defining its Julia set $\mathcal{J}_f$
- then finding the g for which

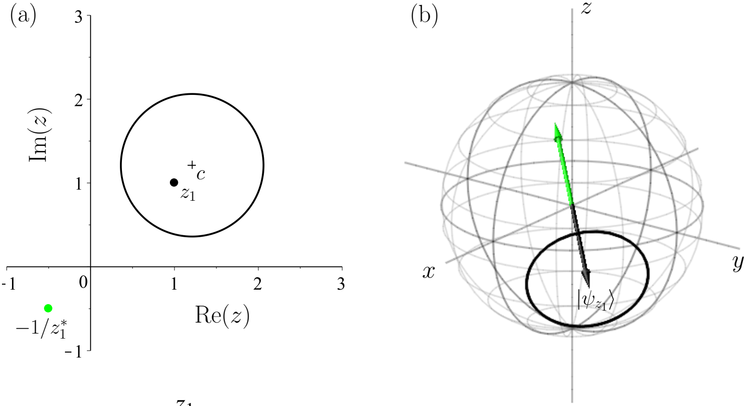
  $$g^{-1}\left(\left\{\mathcal{J}_f\right\}\right) = \mathcal{J}_{f_0}$$
- $f = g \circ f_0 \circ g^{-1}$

Idea of q-state matching

- defining reference state $\left|\psi_{z_1}\right\rangle$
- defining its neighborhood
- then finding implementation of f
- iteration of f realizes q-state matching to $\left|\psi_{z_1}\right\rangle$

O. Kálmán and T. Kiss, arXiv:1708.03610 (2017)

Quantum state matching II



$$c = \frac{z_1}{|s|^2\left(1+|z_1|^2\right) - |z_1|^2}$$

$$r = \frac{|s|\left(1+|z_1|^2\right)\sqrt{1-|s|^2}}{\left||s|^2\left(1+|z_1|^2\right) - |z_1|^2\right|}$$

$$s = \langle \psi_{z_1} \mid \psi_z \rangle$$

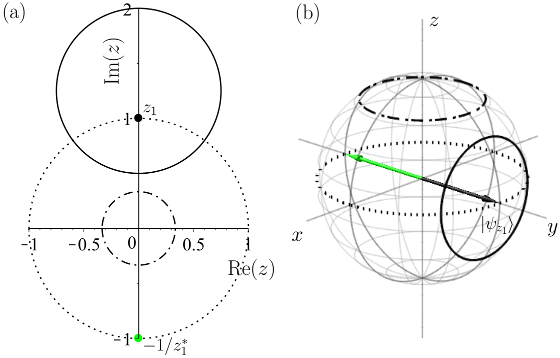
quantum state matching

- reference state: $|\psi_{z_1}\rangle$
- neighborhood: $|s_\varepsilon| = \left|\langle \psi_{z_1} \mid \psi_z \rangle\right|_{\min}$
- if $|s| > |s_\varepsilon|$ then $|\psi_z\rangle \to |\psi_{z_1}\rangle$
- if $|s| < |s_\varepsilon|$ then $|\psi_z\rangle \to |\psi_{-1/z_1^*}\rangle$

Determination of the nonlinear map f for QSM

- The Möbius transformation g can be written as $g = g_{U_{z_1}} \circ g_\varepsilon$
  - $g_\varepsilon(z) = \varepsilon z \qquad \varepsilon = |\varepsilon|\, e^{i\alpha_\varepsilon} \qquad |\varepsilon| = \frac{\sqrt{1-|s_\varepsilon|^2}}{|s_\varepsilon|}$ contracting Möbius
  - $g_{U_{z_1}}(z) = \dfrac{e^{i\alpha_u} z + z_1 e^{-i\alpha_u}}{-z_1^* e^{i\alpha_u} z + e^{-i\alpha_u}}$ unitary Möbius

$$\Rightarrow \quad f(z) = g \circ f_0 \circ g^{-1} = g_{U_{z_1}} \circ g_\varepsilon \circ f_0 \circ g_\varepsilon^{-1} \circ g_{U_{z_1}}^{-1}$$



$z_1 = i$
$\varepsilon = 1/3$
$|s_\varepsilon|^2 = 0.9$

Realization of maps suitable for q-state matching

$$f(z) = \frac{\left(\varepsilon z_1^* |z_1|^2 + 1\right) z^2 + 2z_1 \left(\varepsilon z_1^* - 1\right) z + z_1 (\varepsilon + z_1)}{z_1^* \left(\varepsilon z_1^* - 1\right) z^2 + 2z_1^* (\varepsilon + z_1) z + \varepsilon - z_1 |z_1|^2} \quad (0 < \varepsilon < 1)$$

Direct approach

- two-qubit unitary + a properly defined post-selection protocol
  - e.g. measure whether the state of qubit B is $|0\rangle_B \Rightarrow$ keep qubit A

$$|\psi_1\rangle_A = \frac{1}{\mathcal{N}} \left[|0\rangle_A + \frac{u_{31} + (u_{32} + u_{33}) z + u_{34} z^2}{u_{11} + (u_{12} + u_{13}) z + u_{14} z^2} |1\rangle_A \right]$$

  - require $|\psi_1\rangle_A$ to be

$$|\psi_1\rangle_A = |0\rangle_A + f(z) |1\rangle_A \quad \text{where} \quad f(z) = \frac{a_0 z^2 + a_1 z + a_2}{b_0 z^2 + b_1 z + b_2}$$

  - U can be determined

A. Gilyén, T. Kiss and I. Jex, Sci. Rep. **6**, 20076 (2016).

Realization of maps suitable for q-state matching

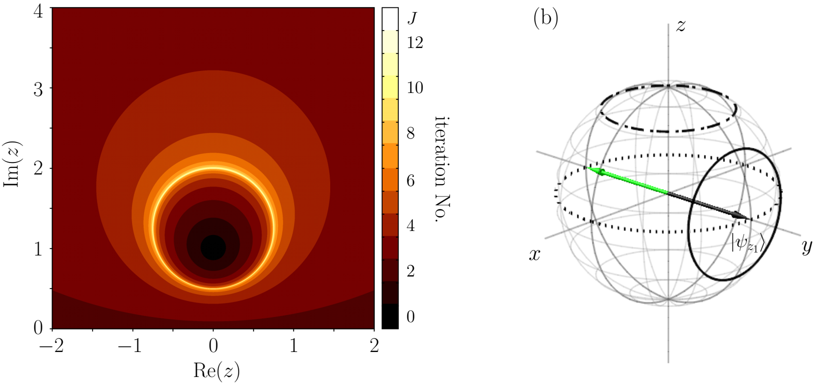
$$f(z) = g_{U_{z_1}} \circ g_\varepsilon \circ f_0 \circ g_\varepsilon^{-1} \circ g_{U_{z_1}}^{-1} = g_{U_{z_1}} \circ f_\varepsilon \circ g_{U_{z_1}}^{-1}$$

Single-qubit gate + a special two-qubit gate

- realize $g_{U_{z_1}}$ by a single-qubit unitary (rotation)
- determine the two-qubit unitary U_ε which realizes $f_\varepsilon = \dfrac{z^2}{\varepsilon}$ $(|\varepsilon| < 1)$

$$U_\varepsilon = \begin{pmatrix} \varepsilon & \frac{1}{\sqrt{2}}\sqrt{1-\varepsilon^2} & -\frac{1}{\sqrt{2}}\sqrt{1-\varepsilon^2} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \\ \sqrt{1-\varepsilon^2} & -\frac{1}{\sqrt{2}}\varepsilon & \frac{1}{\sqrt{2}}\varepsilon & 0 \end{pmatrix}$$

Example: matching with $|\psi_{z_1}\rangle = (|0\rangle + i\,|1\rangle)\,/\,\sqrt{2}$



- f matches qubit states with the state $|\psi_{z_1}\rangle = (|0\rangle + i\,|1\rangle)\,/\,\sqrt{2}$ if they have an initial overlap larger than $|s_\varepsilon|^2 = 0.9$ ($\varepsilon = 1/3$)
- we say a state is matched if after iteration an overlap larger than $|s|^2 = 0.994$ is reached

Conclusion

advantage: a single setup may be reused for the operations
disadvantage: many qubits are lost during the iterations (probabilistic)

- if production of identical pure states + storing of qubits is easy
- nonlinear protocols could be useful for quantum informational tasks
  - quantum state discrimination among sets of states
  - matching an unknown state (e.g. output of a quantum operation) to a desired reference state
    - if qubits of the ensemble have been matched to the desired state, they can be used for further quantum computation ("quantum state error correction")