Exceptional orthogonal polynomials and quasi-invariance

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Bochner's theorem (1929)

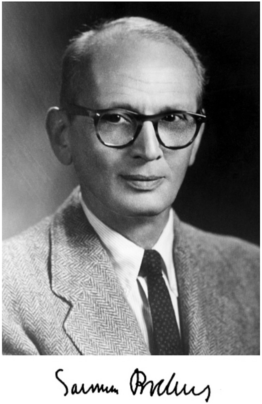
Suppose $p_n(x) \in \mathbb{R}[x]$, $n \in \mathbb{Z}_{\geq 0}$, with $\deg p_n = n$ satisfy

$$A_2(x)\frac{d^2 p_n}{dx^2} + A_1(x)\frac{dp_n}{dx} + A_0(x)p_n = E_n p_n(x).$$

Then $A_j(x) \in \mathbb{R}[x]$ with $\deg A_j \leq j$.
If, in addition,

$$\int_a^b p_m(x)p_n(x)w(x)dx = \delta_{mn}g_n,$$

with $w(x) > 0$, then (up to $x \to ax + b$) the $p_n(x)$ are classical Hermite, Laguerre or Jacobi polynomials.



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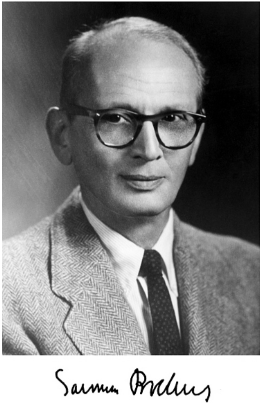

Hermite polynomials

The classical Hermite polynomials

$$H_n(z) = (-1)^n e^{z^2} \frac{d^n}{dz^n} e^{-z^2} = n! \sum_{m=0}^{\lfloor n/2 \rfloor} \frac{(-1)^m}{m!(n-2m)!} (2x)^{n-2m}$$

satisfy

$$\frac{d^2 H_n}{dz^2} - 2z \frac{dH_n}{dz} = -2nH_n$$

and

$$\int_{-\infty}^{\infty} H_n(z) H_m(z) e^{-z^2} dz = \delta_{nm} 2^n n! \sqrt{\pi}.$$

Exceptional orthogonal polynomials

Let $S \subset \mathbb{Z}_{\geq 0}$ be such that $|\mathbb{Z}_{\geq 0} \setminus S| < \infty$.

Gómez-Ullate, Kamran and Milson (2010) called $p_n(x) \in \mathbb{R}[x]$, $n \in S$, a system of exceptional orthogonal polynomials if the following conditions are satisfied.

Eigenvalue equation:

$$A_2(x)\frac{d^2 p_n}{dx^2} + A_1(x)\frac{dp_n}{dx} + A_0(x)p_n = E_n p_n(x).$$

Orthogonality:

$$\int_a^b p_n(x)p_m(x)w(x)dx = \delta_{nm}g_n.$$

Density: $U := \langle p_n : n \in S\rangle$ is dense in $\mathbb{R}[x]$, i.e.

$$(p, p_n) = 0 \ \forall n \in S \ \Rightarrow \ p \equiv 0.$$

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Exceptional Hermite polynomials

Let $\lambda = (\lambda_1, \ldots, \lambda_l)$ be a double partition:

$$\lambda = \mu^2 = (\mu_1, \mu_1, \mu_2, \mu_2, \ldots, \mu_r, \mu_r), \quad 2r = l,$$

and

$$k_j = \lambda_j + l - j, \quad j = 1, \ldots, l.$$

Gómez-Ullate, Kamran and Milson (2014) showed that

$$H_{\lambda,n}(z) := \mathrm{Wr}(H_n, H_{k_1}, \ldots, H_{k_l}), \quad n \in \mathbb{Z}_{\geq 0} \setminus \{k_1, \ldots, k_l\},$$

yield a system of exceptional orthogonal polynomials.

In particular,

$$\int_{-\infty}^{\infty} H_{\lambda,n}(z) H_{\lambda,m}(z) \frac{e^{-z^2}}{W_\lambda(z)^2} = \delta_{nm} \sqrt{\pi} 2^n n! \prod_{i=1}^{l} (n - k_i)(n - k_i - 1),$$

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Singular weight functions

Why restrict attention to double partitions?

Crum (1954) and Adler (1994): The Wronskian $W_\lambda(z)$ has no zeros on $\mathbb{R}$ iff

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The simples non-trivial example: $\lambda = (1)$

The first few polynomials $H_{(1),n} = \mathrm{Wr}(H_n, H_1)$ are

$$\begin{array}{ll} H_{(1),0}(z) = 2 & \text{degree 1 missing} \\ H_{(1),2}(z) = -(2+4z^2) & H_{(1),3}(z) = -16z^3 \\ H_{(1),4}(z) = 12(1+4z^2-4z^4) & H_{(1),5}(z) = 64z^3(5-2z^2) \end{array}$$

Eigenvalue equation:

$$\frac{d^2 H_{(1),n}}{dz^2} - 2\left(z + \frac{1}{z}\right)\frac{dH_{(1),n}}{dz} = -2(1+n)H_{(1),n}.$$

Orthogonality:

$$\int_{i\xi+\mathbb{R}} H_{(1),n}(z)H_{(1),m}(z)\frac{e^{-z^2}}{(2z)^2}\,dz = \delta_{nm}2^n n!\sqrt{\pi}\,2(n-1), \quad \xi \neq 0.$$

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Darboux transformations

Harmonic oscillator:

$$\mathcal{H} := -\frac{d^2}{dz^2} + z^2$$

has eigenfunctions

$$\psi_n(z) = H_n(z)e^{-z^2}, \quad n \in \mathbb{Z}_{\geq 0},$$

w/ eigenvalues $E_n = 2n + 1$.

Letting

$$D_k = \frac{d}{dz} - \frac{\psi_k'}{\psi_k},$$

we get

$$\begin{aligned} D_k^* D_k &= \left(-\frac{d}{dz} - \frac{\psi_k'}{\psi_k}\right)\left(\frac{d}{dz} - \frac{\psi_k'}{\psi_k}\right) \\ &= -\frac{d^2}{dz^2} + \frac{\psi_k''}{\psi_k} \\ &= -\frac{d^2}{dz^2} + z^2 - E_k, \end{aligned}$$

so that

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Reversing order,

$$D_k D_k^* = \left(\frac{d}{dz} - \frac{\psi_k'}{\psi_k}\right)\left(-\frac{d}{dz} - \frac{\psi_k'}{\psi_k}\right)$$
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gives

$$\mathcal{H}_k := D_k D_k^* + E_k = -\frac{d^2}{dz^2} + z^2 - 2\left(\frac{\psi_k'}{\psi_k}\right)'.$$

Introducing

$$\psi_{k,n} := D_k \psi_n = \frac{\mathrm{Wr}(\psi_n, \psi_k)}{\psi_k}, \quad n \neq k,$$

we obtain

$$\mathcal{H}_k \psi_{k,n} = D_k(D_k^* D_k + E_k)\psi_n = D_k \mathcal{H} \psi_n = E_n \psi_{k,n}.$$

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In particular,

$$\mathcal{H}_1 = -\frac{d^2}{dz^2} + z^2 + \frac{2}{z^2}$$

and

$$\begin{aligned}\psi_{1,n}(z) &= \frac{\mathrm{Wr}\big(H_n(z)e^{-z^2/2}, H_1(z)e^{-z^2/2}\big)}{H_1(z)e^{-z^2/2}} \\ &= \frac{\mathrm{Wr}\big(H_n(z), H_1(z)\big)}{H_1(z)} e^{-z^2/2} \\ &= H_{(1),n}(z)\frac{e^{-z^2/2}}{2z}.\end{aligned}$$

Orthogonality

$$\int_{i\xi+\mathbb{R}} H_{(1),n}(z)H_{(1),m}(z)\frac{e^{-z^2}}{(2z)^2}dz = \int_{i\xi+\mathbb{R}} \psi_{1,n}(z)\psi_{1,m}(z)dz$$

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Quasi-invariance

Note that

$$\frac{d}{dz}(z\psi_{1,n}) = \frac{d}{dz}(H_n' H_1 - H_n H_1')\frac{e^{-z^2/2}}{2}$$
$$= \frac{d}{dz}(zH_n' - H_n)e^{-z^2/2}$$
$$= z(\cdots)e^{-z^2/2} = 0 \quad \text{at } z = 0,$$

i.e.

$$\psi_{1,n}(-z) = (-1)\psi_{1,n}(z) + o(z).$$

More generally, we say that a meromorphic function $\psi(z)$ is quasi-invariant at $z = z_i$ w/ multiplicity $m_i \in \mathbb{Z}_{\geq 0}$ if

1. $\psi(z)(z - z_i)^{m_i}$ is analytic at $z = z_i$,
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Quasi-invariance

Let

$$\mathcal{Q}_{(1)} = \left\{ p \in \mathbb{C}[z] : \psi(z) = p(z)\frac{e^{-z^2/2}}{2z} \text{ q.} - \text{inv. at } z = 0 \text{ w/ mult. } 1 \right\}.$$

Proposition

We have

$$\mathbb{C}\langle H_{(1),n} : n \neq 1 \rangle = \mathcal{Q}_{(1)}.$$

Proof.

The subspaces have the same codimension in $\mathbb{C}[z]$, since

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Quasi-invariance

Let

$$\mathcal{Q}_{(1)} = \left\{ p \in \mathbb{C}[z] : \psi(z) = p(z)\frac{e^{-z^2/2}}{2z} \text{ q.} - \text{inv. at } z = 0 \text{ w/ mult. } 1 \right\}.$$

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We have

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A Hermitian product

Definition

Assuming $\xi \in \mathbb{R} \setminus \{0\}$, we set

$$\langle p, q \rangle = \int_{i\xi+\mathbb{R}} p(z)\bar{q}(z)\frac{e^{-z^2}}{(2z)^2}\,dz, \quad p, q \in \mathcal{Q}_{(1)},$$

where $\bar{q}(z) = \overline{q(\bar{z})}$.

Lemma

$\langle \cdot, \cdot \rangle$ does not depend on the value of ξ.

Proof.

By the residue thm and quasi-invariance, we have

$$\begin{aligned}\langle p, q \rangle_\xi &= \langle p, q \rangle_{-\xi} + \frac{d}{dz}\left(z^2 p(z)\bar{q}(z)\frac{e^{-z^2}}{(2z)^2} \right)\Big|_{z=0} \\ &= \langle p, q \rangle_{-\xi} + \frac{d}{dz}\left(\left[z \cdot p(z)\frac{e^{-z^2/2}}{2z} \right] \cdot \left[z \cdot \bar{q}(z)\frac{e^{-z^2/2}}{2z} \right] \right)\Big|_{z=0} \\ &= \langle p, q \rangle_{-\xi}.\end{aligned}$$

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Proposition

$\langle\cdot,\cdot\rangle$ *is Hermitian.*

Proof.

Introducing

$$w(z) = \frac{e^{-z^2}}{(2z)^2}$$

and observing $\bar{w}(z) = w(z)$, we deduce

$$\begin{aligned}\langle p, q\rangle_\xi &= \overline{\int_{\mathbb{R}} p(i\xi + x)\bar{q}(i\xi + x)w(i\xi + x)dx} \\ &= \overline{\int_{\mathbb{R}} \bar{p}(-i\xi + x)q(-i\xi + x)w(-i\xi + x)dx} \\ &= \overline{\langle q, p\rangle_{-\xi}}.\end{aligned}$$

Hence, the assertion follows from the lemma. □

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Main result

Theorem

The Exceptional Hermite Polynomials $H_{(1),n}$, $n \neq 1$, constitute an orthogonal basis in $\mathcal{Q}_{(1)}$ w/ respect to the Hermitian product $\langle \cdot, \cdot \rangle$. Specifically,

$$\langle H_{(1),n}, H_{(1),m} \rangle = \delta_{nm} 2^n n! \sqrt{\pi}\, 2(n-1).$$

This result generalises to Exceptional Hermite Polynomials labelled by an arbitrary partition λ, see

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A multivariate generalisation

The quantum rational (A_{n-1}) Calogero-Moser system

Can be defined by the Hamiltonian

$$H_n = -\sum_{j=1}^{n} \frac{\partial^2}{\partial z_j^2} + 2m(m-1) \sum_{1 \le j < k \le n} \frac{1}{(z_j - z_k)^2},$$

where $N \in \mathbb{N}$ (particle number) and $m > 0$ (coupling parameter).

Associated integrable system (n commuting PDOs):

$$H_n^1 = -i \sum_{j=1}^{n} \frac{\partial}{\partial z_j}, \quad H_n,$$

$$H_n^k = (-i)^k \sum_{j=1}^{n} \frac{\partial^k}{\partial z_j^k} + \cdots, \quad k = 3, \ldots, n,$$

(*Olshanetsky & Perelomov (1983), Ujino, Hikami & Wadati (1992), and others*).

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A_{n-1} quasi-invariants

Let $m \in \mathbb{Z}_{\geq 0}$.

A polynomial $q \in \mathbb{C}[x]$ is said to be m-quasi-invariant if

$$\left(\frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_j}\right) q(z) = \left(\frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_j}\right)^3 q(z) = \cdots = \left(\frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_j}\right)^{2m-1} q(z) \equiv 0$$

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Such polynomials form an algebra, denoted $\mathcal{Q}_m$.

The $m \in \mathbb{Z}_{\geq 0}$ Calogero-Moser system is algebraically integrable: for each $q \in \mathcal{Q}_m$ exists PDO

$$H_q = q\left(-i\partial/\partial z_1, \ldots, -i\partial/\partial z_n\right) + \cdots$$

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$$H_{z_1^2 + \cdots + z_n^2} = H_n, \quad [H_q, H_p] = 0, \; \forall q, p \in \mathcal{Q}_m,$$

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Bilinear forms on $\mathcal{Q}_m$

For $q \in \mathcal{Q}_m$, let

$$L_q = A_m(z) \circ H_q \circ A_m(z)^{-1}, \quad A_m(z) = \prod_{1 \le i < j \le n} (z_i - z_j)^m.$$

Canonical:

$$\langle q, p \rangle := (L_q p)(0).$$

'Hermite-like':

$$(q, p) := \int_{i\xi + \mathbb{R}^n} \frac{q(z)p(z)}{A_m(z)^2} e^{-z^2} \, dz.$$

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A result in progress

Feigin, H. & Veselov: There exists a degree-preserving operator of the form

$$H_\bullet : \mathcal{Q}_m \to \mathcal{Q}_m, q \mapsto \langle F(z, \bullet), q \rangle$$

such that

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$$\left(L_{z_1^2 + \cdots + z_n^2} - 2 \sum_{i=1}^{n} z_i \frac{\partial}{\partial z_i} \right) H_q = -2(\deg q) H_q$$

whenever q is homogeneous.

The (generating) function F is given by

$$F(z, w) = \phi(z, w) \exp\left(-w^2\right),$$

w/ ϕ the so-called (A_{n-1}) rational Baker-Akhiezer function.

The result extends to all Coxeter groups.

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