Quantum-classical dualities in integrable systems

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Plan of the talk:

1. Algebraic Bethe ansatz for XXX spin chain
2. Classical Ruijsenaars-Schneider model
3. Duality
4. Quantum-quantum version of duality
5. Classical-classical version of duality
6. Further generalizations

Algebraic Bethe ansatz

1. Exchange relations (or RTT- or RLL- relations):

$$R^{\eta}_{00'}(z,w)(T(z)\otimes I)(I\otimes T(w)) = (I\otimes T(w))(T(z)\otimes I)R^{\eta}_{00'}(z,w)\,,$$

$$R_{00'}(z,w)T_0(z)T_{0'}(w) = T_{0'}(w)T_0(z)R_{00'}(z,w)\,,$$

T is a monodromy matrix – operator-valued matrix in Mat$(N,\mathbb{C})$, and $R_{00'}(z,w)$ is an invertible matrix from Mat$(N,\mathbb{C})^{\otimes 2}$ with $\mathbb{C}$-valued matrix elements depending on spectral parameters z,w and the Planck constant η.

Example:

$$T(z) = \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix}\,,$$

where $A(z), B(z), C(c), D(z)$ are some operators.

$$R_{00'}(z-w) = \begin{pmatrix} 1+\frac{\eta}{z-w} & 0 & 0 & 0 \\ 0 & 1 & \frac{\eta}{z-w} & 0 \\ 0 & \frac{\eta}{z-w} & 1 & 0 \\ 0 & 0 & 0 & 1+\frac{\eta}{z-w} \end{pmatrix}$$

$$(T(z) \otimes I) = \begin{pmatrix} A(z) & 0 & B(z) & 0 \\ 0 & A(z) & 0 & B(z) \\ C(z) & 0 & D(z) & 0 \\ 0 & C(z) & 0 & D(z) \end{pmatrix}$$

$$(I \otimes T(w)) = \begin{pmatrix} A(w) & B(w) & 0 & 0 \\ C(w) & D(w) & 0 & 0 \\ 0 & 0 & A(w) & B(w) \\ 0 & 0 & C(w) & D(w) \end{pmatrix}$$

$R^{\eta}_{00'}(z,w)(T(z) \otimes I)(I \otimes T(w)) = (I \otimes T(w))(T(z) \otimes I)R^{\eta}_{00'}(z,w)$, We get 16 commutation relations:

$$A(w)B(z) = (1 + \tfrac{\eta}{z-w})B(z)A(w) - \tfrac{\eta}{z-w}B(w)A(z)$$

$$B(w)A(z) = (1 + \tfrac{\eta}{z-w})A(z)B(w) - \tfrac{\eta}{z-w}A(w)B(z)$$

.....

1. Exchange relations (or RTT- or RLL- relations):

$$R^{\eta}_{00'}(z,w)(T(z)\otimes I)(I\otimes T(w)) = (I\otimes T(w))(T(z)\otimes I)R^{\eta}_{00'}(z,w)\,,$$

$$R_{00'}(z,w)T_0(z)T_{0'}(w) = T_{0'}(w)T_0(z)R_{00'}(z,w)\,,$$

T is a monodromy matrix – operator-valued matrix in Mat$(N,\mathbb{C})$, and $R_{00'}(z,w)$ is an invertible matrix from Mat$(N,\mathbb{C})^{\otimes 2}$ with $\mathbb{C}$-valued matrix elements depending on spectral parameters z,w and the Planck constant η.

2. The consistency condition for triple product $T_1(z_1)T_2(z_2)T_3(z_3)$ (123→321 in two ways) is the quantum Yang-Baxter equation:

$$R^{\eta}_{12}(z_{12})R^{\eta}_{13}(z_{13})R^{\eta}_{23}(z_{23}) = R^{\eta}_{23}(z_{23})R^{\eta}_{13}(z_{13})R^{\eta}_{12}(z_{12})\,, \quad z_{ij}=z_i-z_j$$

For example, the rational Yang's R-matrix

$$R^{\eta}_{00'}(z-w) = I\otimes I + \frac{\eta}{z-w}P_{00'}\,, \quad P_{00'} = \sum_{i,j} E_{ij}\otimes E_{ji}\,.$$

Multiplying RTT-relations by $R_{00'}(z,w)^{-1}$

$$T_0(z)T_{0'}(w) = R_{00'}(z,w)^{-1}T_{0'}(w)T_0(z)R_{00'}(z,w)\,,$$

and taking the trace over the auxiliary spaces 0, 0′ we get

$$[\mathbf{T}(z), \mathbf{T}(w)] = 0$$

where

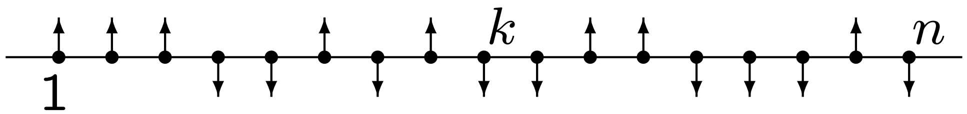
$$\mathbf{T}(z) = \mathrm{tr}\, T(z)\,.$$

is a transfer-matrix. Having an expansion of the form

$$\mathbf{T}(z) = \sum_k (z - z_0)^k I_k$$

we obtain a set of commuting Hamiltonians

$$[I_k, I_m] = 0 \quad \forall k, m\,.$$



Spin chains:

XXX Heisenberg chain

$$H=\sum_{k=1}^{n}\left(\sigma_k^x\sigma_{k+1}^x+\sigma_k^y\sigma_{k+1}^y+\sigma_k^z\sigma_{k+1}^z\right),$$

where the operators $\sigma_k^\alpha = I\otimes\cdots\otimes\sigma^\alpha\otimes\cdots\otimes I,\ \alpha=x,y,z$ and $\sigma_{n+1}^\alpha=\sigma_1^\alpha$ (closed chain). The Hilbert space is $\mathcal{H}=V_1\otimes\ldots\otimes V_n$, For $N=2$ $V_k\cong\mathbb{C}^2$, $\dim\mathcal{H}=2^n$.

$$R_{00'}(z-w)=\begin{pmatrix} 1+\frac{\eta}{z-w} & 0 & 0 & 0\\ 0 & 1 & \frac{\eta}{z-w} & 0\\ 0 & \frac{\eta}{z-w} & 1 & 0\\ 0 & 0 & 0 & 1+\frac{\eta}{z-w}\end{pmatrix}$$

To each site we assign the Lax operator $L_k(z)$:

$$L_k(z)=\begin{pmatrix} I+\frac{\eta}{z}\sigma_k^z & \frac{\eta}{z}\sigma_k^-\\ \frac{\eta}{z}\sigma_k^+ & I-\frac{\eta}{z}\sigma_k^z\end{pmatrix}=R_{0k}(z)\in \mathrm{Mat}(2,\mathbb{C})\otimes\mathrm{End}(\mathcal{H})\cong\mathrm{Mat}(2,\mathbb{C})^{\otimes(n+1)}$$

$$T(z) = \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix},$$

where $A(z), B(z), C(c), D(z)$ are some operators, is the product of Lax operators

$$T(z) = L_n(z) \dots L_2(z) L_1(z).$$

If $L_k(z)$ satisfy the RLL relations

$$R(z-w)\Big(L_k(z)\otimes I\Big)\Big(I\otimes L_k(w)\Big) = \Big(I\otimes L_k(w)\Big)\Big(L_k(z)\otimes I\Big)R(z-w).$$

then $T(z)$ satisfies RLL relations as well.
(in our case it is true $R_{00'}(z-w)R_{0k}(z)R_{0'k}(w) = R_{0'k}(w)R_{0k}(z)R_{00'}(z-w)$)

The Hamiltonians are computed from the tranfer-matrix

$$\mathbf{T}(z) = \mathrm{tr}(L_n(z)\dots L_2(z)L_1(z)) = A(z) + D(z)\,.$$

It can be shown that for $\mathbf{t}(z) = z^n \mathbf{T}(z)$ the Heisenberg model is reproduced

$$H = 2\eta \frac{d\mathbf{t}(z)}{dz}\,\mathbf{t}^{-1}(z)\Big|_{z=\frac{\eta}{2}} - n = 2\sum_{k=1}^{n} P_{k\,k+1} - n.$$

Generalized spin chain

Inhomogeneous twisted XXX chain:

$$T(z) = \mathbf{g} L_n(z - q_n) \dots L_1(z - q_1)\,,$$

where $q_1, ..., q_n$ are inhomogeneity parameters and $\mathbf{g} = \sum_{a=1}^{N} E_{aa} g_a$ – (diagonal) constant twist matrix. It comes from the symmetry of R-matrix:

$$R(z-w)\Big(\mathbf{g} \otimes \mathbf{g}\Big) = \Big(\mathbf{g} \otimes \mathbf{g}\Big) R(z-w).$$

The generating function of commuting integrals of motion (the transfer matrix) depending on the spectral parameter z being of the form

$$\mathbf{T}(z) = \mathrm{tr}_0\Big(\mathbf{g}^{(0)} \widetilde{\mathbf{R}}_{0n}(z-q_n) \dots \widetilde{\mathbf{R}}_{02}(z-q_2) \widetilde{\mathbf{R}}_{01}(z-q_1)\Big) = \mathrm{tr}\, \mathbf{g} \cdot \mathbf{I} + \sum_{j=1}^{n} \frac{\eta \mathbf{H}_j}{z - q_j},$$

$$\widetilde{\mathbf{R}}_{ij}(z) = \mathbf{I} + \frac{\eta}{z} \mathbf{P}_{ij}\,, \quad \mathbf{P}_{ij} = \sum_{a,b=1}^{N} E_{ab}^{(i)} E_{ba}^{(j)}\,.$$

Its residues are non-local Hamiltonians:

$$\mathbf{H}_i = \widetilde{\mathbf{R}}_{i\,i-1}(q_i - q_{i-1}) \dots \widetilde{\mathbf{R}}_{i1}(q_i - q_1) \mathbf{g}^{(i)} \widetilde{\mathbf{R}}_{in}(q_i - q_n) \dots \widetilde{\mathbf{R}}_{i\,i+1}(q_i - q_{i+1})$$

Idea of the Bethe ansatz

Write down commutation relations:

$$[T^{jk}(u), T^{jk}(v)] = 0, \qquad j,k = 1,2,$$

$$A(w)B(z) = (1 + \tfrac{\eta}{z-w})B(z)A(w) - \tfrac{\eta}{z-w}B(w)A(z)$$

$$B(w)A(z) = (1 + \tfrac{\eta}{z-w})A(z)B(w) - \tfrac{\eta}{z-w}A(w)B(z)$$

$$\cdots\cdots$$

Recall that the elements of the monodromy matrices act in some Hilbert space $\mathcal{H}$. In the framework of the algebraic Bethe ansatz, very small requirements are imposed on this space. Namely, it is necessary that there exist a vacuum vector such that ($a(z)$, $d(z)$ below are some $\mathbb{C}$-valued functions)

$$A(z)|0\rangle = a(z)|0\rangle, \qquad D(z)|0\rangle = d(z)|0\rangle, \qquad C(z)|0\rangle = 0.$$

The vacuum vector for XXX spin chain: all spins up in $N = 2$ case, i.e.

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 \otimes \cdots \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}_n .$$

We look for the eigenvectors of the transfer-matrix ψ in the form

$$\psi = B(\mu_1)...B(\mu_{n_1})|0\rangle\,, \qquad\qquad (A(z)+D(z))\psi = \lambda(z,\mu)\psi\,.$$

The vacuum vector is the eigenvector for the operators $A(z)$ and $D(z)$, while the operator $C(z)$ annihilates it. The action of the operator $B(z)$ onto the vacuum is free. It is assumed that acting with this operator on vacuum we generate all the space $\mathcal{H}$. The set of parameters $\mu_1,...,\mu_{n_1}$ to be determined – Bethe roots.

Compute the action of the operator $A(z)$ onto the vector ψ:

$$A(z)B(\mu_1)...B(\mu_{n_1})|0\rangle =$$

$$= a(z)\Lambda(z|\mu)B(\mu_1)...B(\mu_{n_1})|0\rangle + \sum_{k=1}^{n_1} a(\mu_k)\Lambda_k(z|\mu)B(z)\prod_{l\neq k} B(\mu_l)|0\rangle.$$

After computing the same for $D(z)$ and summing up we get

$$\mathbf{T}(z)B(\mu_1)...B(\mu_{n_1})|0\rangle = \Big(a(z)\Lambda(v|\mu) + d(v)\tilde{\Lambda}(z|\mu)\Big)B(\mu_1)...B(\mu_{n_1})|0\rangle$$

$$+\sum_{k=1}^{n}\Big(a(\mu_k)\Lambda_k(z|\mu) + d(u_k)\tilde{\Lambda}_k(z|\mu)\Big)B(z)\prod_{l\neq k} B(\mu_l)|0\rangle.$$

Conditions for vanishing of the unwanted terms – **Bethe equations:**

$$a(\mu_k)\Lambda_k(z|\mu) + d(\mu_k)\tilde{\Lambda}_k(z|\mu) = 0, \qquad k = 1, \ldots, n_1$$

The generating function of the eigenvalues of transfer-matrix:

$$\lambda(z|\mu) = a(z)\Lambda(z|\mu) + d(z)\tilde{\Lambda}(z|\mu)\,.$$

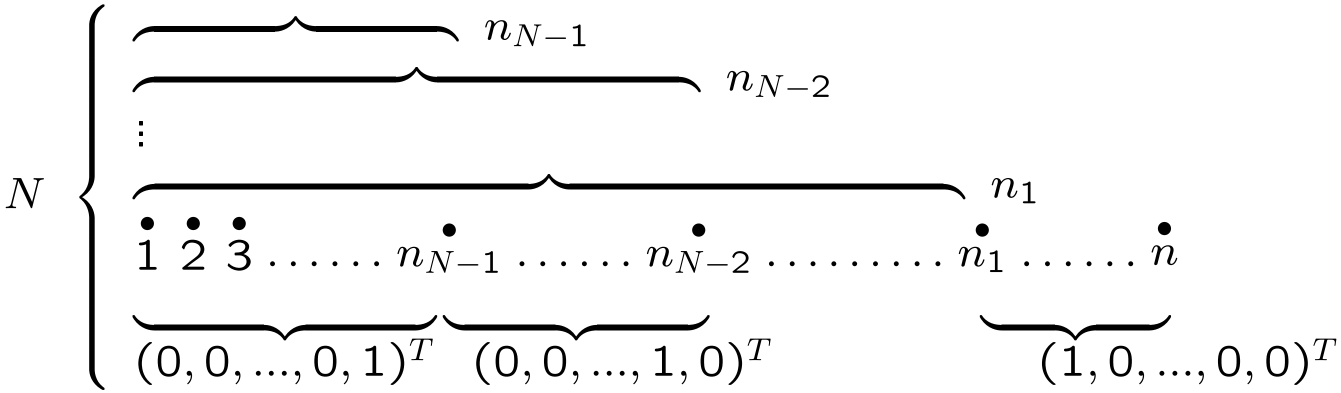
Example the Bethe equations (BE):

$$\frac{g_1}{g_2}\prod_{k=1}^{n}\frac{\mu_\alpha - q_k + \eta}{\mu_\alpha - q_k} = \prod_{\gamma\neq\alpha}^{n_1}\frac{\mu_\alpha - \mu_\gamma + \eta}{\mu_\alpha - \mu_\gamma - \eta}\,, \qquad \alpha = 1\,, ...\,, n_1\,.$$

The eigenvalues:

$$\frac{1}{\eta}H_i^{\mathrm{XXX}} = g_1\prod_{k\neq i}^{n}\frac{q_i - q_k + \eta}{q_i - q_k}\prod_{\gamma=1}^{n_1}\frac{q_i - \mu_\gamma - \eta}{q_i - \mu_\gamma}\,, \qquad i = 1\,, \ldots\,, n,$$

Nested Bethe ansatz (n_a – number of Bethe roots at a-th level)



Eigenvalues H_j of Hamiltonians $\mathbf{H}_j$

$$\frac{1}{\eta}H_i = g_1 \prod_{k=1}^{n} \frac{q_i - q_k + \eta}{q_i - q_k} \prod_{\gamma=1}^{n_1} \frac{q_i - \mu^1_\gamma - \eta}{q_i - \mu^1_\gamma}$$

depend on a solution $\left\{ \{\mu^1_i\}_{n_1}, \ldots, \{\mu^{N-1}_i\}_{n_{N-1}} \right\}$ of Bethe equations:

$$BE_1: \quad g_1 \prod_{k=1}^{n} \frac{\mu^1_\beta - q_k + \eta}{\mu^1_\beta - q_k} = g_2 \prod_{\gamma\neq\beta}^{n_1} \frac{\mu^1_\beta - \mu^1_\gamma + \eta}{\mu^1_\beta - \mu^1_\gamma - \eta} \prod_{\gamma=1}^{n_2} \frac{\mu^1_\beta - \mu^2_\gamma - \eta}{\mu^1_\beta - \mu^2_\gamma}$$

$$BE_b: \quad g_b \prod_{\gamma=1}^{n_{b-1}} \frac{\mu^b_\beta - \mu^{b-1}_\gamma + \eta}{\mu^b_\beta - \mu^{b-1}_\gamma} = g_{b+1} \prod_{\gamma\neq\beta}^{n_b} \frac{\mu^b_\beta - \mu^b_\gamma + \eta}{\mu^b_\beta - \mu^b_\gamma - \eta} \prod_{\gamma=1}^{n_{b+1}} \frac{\mu^b_\beta - \mu^{b+1}_\gamma - \eta}{\mu^b_\beta - \mu^{b+1}_\gamma}$$

Classical Ruijsenaars-Schneider model: $H^{\mathsf{RS}} = \sum_{j=1}^{n} e^{\varepsilon p_j} \prod_{k\neq j}^{n} \frac{q_j - q_k + \nu\varepsilon}{q_j - q_k}$

$$\dot{q}_j = \frac{\partial H^{\mathsf{RS}}}{\partial p_i} = \varepsilon e^{\varepsilon p_j} \prod_{k\neq j}^{N} \frac{q_j - q_k + \nu\varepsilon}{q_j - q_k}\,, \qquad H^{\mathsf{RS}} = \mathrm{tr}\, L^{\mathsf{RS}}\,,$$

$$L^{\mathsf{RS}}_{ij}(\{\dot{q}_i\}_n, \{q_i\}_n, \hbar) = \frac{\nu\varepsilon\, e^{\varepsilon p_j}}{q_i - q_j + \nu\varepsilon} \prod_{k\neq j}^{n} \frac{q_j - q_k + \nu\varepsilon}{q_j - q_k} = \frac{\nu\, \dot{q}_j}{q_i - q_j + \nu\varepsilon}$$

$$L^{\mathsf{RS}} = \frac{1}{\varepsilon} \begin{pmatrix} \dot{q}_1 & \frac{\eta\,\dot{q}_2}{q_1 - q_2 + \eta} & \frac{\eta\,\dot{q}_3}{q_1 - q_3 + \eta} & \dots & \frac{\eta\,\dot{q}_n}{q_1 - q_n + \eta} \\ \frac{\eta\,\dot{q}_1}{q_2 - q_1 + \eta} & \dot{q}_2 & \frac{\eta\,\dot{q}_3}{q_2 - q_3 + \eta} & \dots & \frac{\eta\,\dot{q}_n}{q_2 - q_n + \eta} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\eta\,\dot{q}_1}{q_n - q_1 + \eta} & \frac{\eta\,\dot{q}_2}{q_n - q_2 + \eta} & \frac{\eta\,\dot{q}_3}{q_n - q_3 + \eta} & \dots & \dot{q}_n \end{pmatrix}, \qquad \eta = \nu\varepsilon$$

Factorization of the Ruijsenaars-Schneider Lax matrix:

Introduce the diagonal matrices D_0 and $D_\hbar$

$$(D_0)_{ij}(\mathbf{u}_K) = \delta_{ij} \prod_{k\neq i}^{K} (u_i - u_k)\,,$$

$$(D_\hbar)_{ij}(\mathbf{u}_K) = \delta_{ij} \prod_{k\neq i}^{K} (u_i - u_k + \hbar)\,,$$

$$i\,,j = 1\,,\ldots\,,K\,,$$

the Vandermonde matrix $V_{ij}(\mathbf{u}_K) = u_j^{i-1}$, $i\,,j = 1\,,\ldots\,,K$, and the triangular matrix

$$(C_{\hbar,K})_{ij} = \begin{cases} \dfrac{(i-1)!\,\hbar^{i-j}}{(j-1)!(i-j)!}\,, & j \leq i\,, \\ 0\,, & j > i\,, \end{cases}$$

$$V(u + \mathbf{e}_K\hbar) = C_{\hbar,K} V(u)$$
$$\mathbf{e}_K = (1, \ldots, 1)$$

Then the following matrix has all eigenvalues equal to g:
(due to $\det(L-\lambda) = \det(C_\hbar - \lambda)$)

$$L_{ij}(\mathbf{x}_N, g) = \frac{g\,\hbar}{x_i - x_j + \hbar} \prod_{k\neq j}^{N} \frac{x_j - x_k + \hbar}{x_j - x_k}\,, \quad i\,, j = 1\,, ...\,, N$$

$$L(\mathbf{x}_N, g) = g\, D_\hbar^{-1}(\mathbf{x}_N)\, V^T(\mathbf{x}_N + \hbar\,\mathbf{e}_N)\, \left(V^T\right)^{-1}(\mathbf{x}_N)\, D_\hbar(\mathbf{x}_N) =$$

$$= g\, D_\hbar^{-1}(\mathbf{x}_N)\, V^T(\mathbf{x}_N)\, C^T_{\hbar,N}\, \left(V^T\right)^{-1}(\mathbf{x}_N)\, D_\hbar(\mathbf{x}_N)\,, \quad \mathbf{e}_N = (1, ..., 1)$$

and similarly for

$$\widetilde{L}_{\alpha\beta}(\mathbf{y}_M, g) = \frac{g\,\hbar}{y_\alpha - y_\beta + \hbar} \prod_{\gamma\neq\beta}^{M} \frac{y_\beta - y_\gamma - \hbar}{y_\beta - y_\gamma}\,, \quad \alpha\,, \beta = 1\,, ...\,, M\,.$$

$$\widetilde{L}(\mathbf{y}_M, g) = g\, D_0(\mathbf{y}_M)\, V^{-1}(\mathbf{y}_M)\, V(\mathbf{y}_M - \hbar\,\mathbf{e}_M)\, D_0^{-1}(\mathbf{y}_M) =$$

$$= g\, D_0(\mathbf{y}_M)\, V^{-1}(\mathbf{y}_M)\, C_{-\hbar,M}\, V(\mathbf{y}_M)\, D_0^{-1}(\mathbf{y}_M)\,.$$

Quantum-classical duality

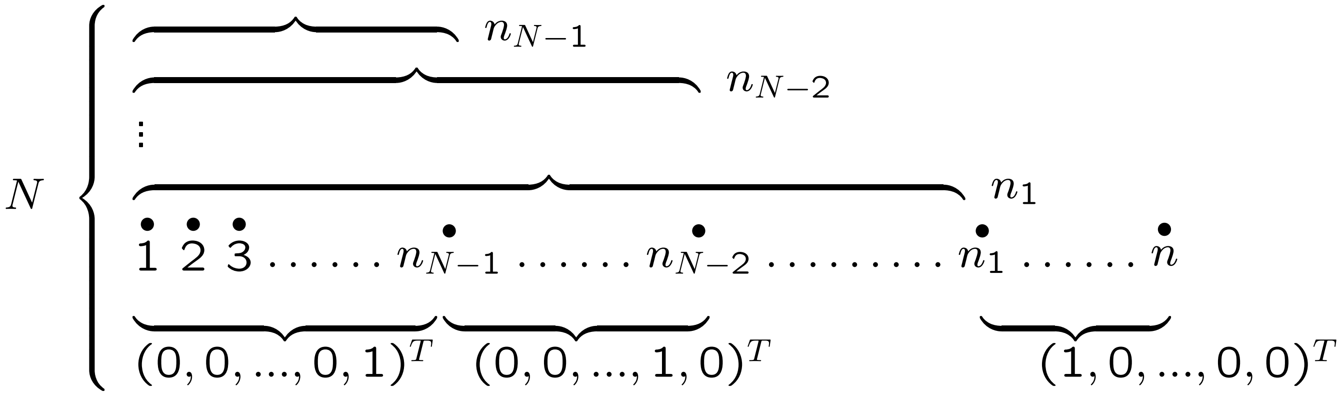
On the quantum side: consider the inhomogeneous $GL(N)$ generalized spin chain of XXX type on n sites with inhomogeneity parameters q_i and vector representations at each site. Let us impose twisted boundary conditions with the twist matrix $\mathbf{g} = \mathrm{diag}(g_1, \ldots, g_N)$, with the generating function of commuting integrals of motion (the transfer matrix) depending on the spectral parameter z being of the form

$$\mathbf{T}(z) = \mathrm{tr}_0\Big(\mathbf{g}^{(0)}\widetilde{\mathbf{R}}_{0n}(z-q_n)\ldots\widetilde{\mathbf{R}}_{02}(z-q_2)\widetilde{\mathbf{R}}_{01}(z-q_1)\Big) = \mathrm{tr}\,\mathbf{g}\cdot\mathbf{I} + \sum_{j=1}^{n}\frac{\eta\mathbf{H}_j}{z-q_j},$$

$$\widetilde{\mathbf{R}}_{ij}(z) = \mathbf{I} + \frac{\eta}{z}\mathbf{P}_{ij}\,, \quad \mathbf{P}_{ij} = \sum_{a,b=1}^{N} E_{ab}^{(i)}E_{ba}^{(j)}\,.$$

Its residues are (non-local) Hamiltonians

$$\mathbf{H}_i = \widetilde{\mathbf{R}}_{i\,i-1}(q_i-q_{i-1})\ldots\widetilde{\mathbf{R}}_{i1}(q_i-q_1)\mathbf{g}^{(i)}\widetilde{\mathbf{R}}_{in}(q_i-q_n)\ldots\widetilde{\mathbf{R}}_{i\,i+1}(q_i-q_{i+1})$$



Eigenvalues H_j of Hamiltonians $\mathbf{H}_j$

$$\frac{1}{\eta} H_i = g_1 \prod_{k=1}^{n} \frac{q_i - q_k + \eta}{q_i - q_k} \prod_{\gamma=1}^{n_1} \frac{q_i - \mu_\gamma^1 - \eta}{q_i - \mu_\gamma^1}$$

depend on a solution $\left\{ \{\mu_i^1\}_{n_1}, \ldots, \{\mu_i^{N-1}\}_{n_{N-1}} \right\}$ of Bethe equations:

$$BE_1: \quad g_1 \prod_{k=1}^{n} \frac{\mu_\beta^1 - q_k + \eta}{\mu_\beta^1 - q_k} = g_2 \prod_{\gamma \neq \beta}^{n_1} \frac{\mu_\beta^1 - \mu_\gamma^1 + \eta}{\mu_\beta^1 - \mu_\gamma^1 - \eta} \prod_{\gamma=1}^{n_2} \frac{\mu_\beta^1 - \mu_\gamma^2 - \eta}{\mu_\beta^1 - \mu_\gamma^2}, \ \beta = 1 \ldots n_1$$

$$BE_b: \quad g_b \prod_{\gamma=1}^{n_{b-1}} \frac{\mu_\beta^b - \mu_\gamma^{b-1} + \eta}{\mu_\beta^b - \mu_\gamma^{b-1}} = g_{b+1} \prod_{\gamma \neq \beta}^{n_b} \frac{\mu_\beta^b - \mu_\gamma^b + \eta}{\mu_\beta^b - \mu_\gamma^b - \eta} \prod_{\gamma=1}^{n_{b+1}} \frac{\mu_\beta^b - \mu_\gamma^{b+1} - \eta}{\mu_\beta^b - \mu_\gamma^{b+1}}, \ \beta = 1 \ldots n_b$$

The claim is that under the substitution:

1. η is the Planck constant
2. positions of RS particles q_i are inhomogeneous parameters
3. the velocities are eigenvalues of quantum Hamitonians

$$\dot{q}_j = \frac{1}{\eta} H_j\Big(\{q_i\}_n; \{\mu_i^1\}_{n_1}, \ldots, \{\mu_i^{N-1}\}_{n_{N-1}}\Big), \quad j = 1, ..., n, \tag{1}$$

where the set of μ_i^a's is any solution of the nested BE for the spin chain, the eigenvalues of the Lax matrix are

$$\Big(\underbrace{g_1, \ldots, g_1}_{n-n_1}, \underbrace{g_2, \ldots, g_2}_{n_1-n_2}, \ldots, \underbrace{g_{N-1}, \ldots, g_{N-1}}_{n_{N-2}-n_{N-1}}, \underbrace{g_N, \ldots, g_N}_{n_{N-1}}\Big). \tag{2}$$

i.e.

$$\det\left[L^{\mathrm{RS}}\left(\frac{1}{\eta}\left\{H_j\right\}_n, \left\{q_j\right\}_n, \eta\varepsilon\right)\Big|_{BE} - \lambda\right] = \prod_{a=1}^{N} (g_a - \lambda)^{M_a}, \tag{3}$$

where $M_1 = n - n_1$, $M_a = n_{a-1} - n_a$ $(2 \leq a \leq N)$

On the classical side: consider the RS model with coupling constant η and the number of particles, n, equal to the number of sites of the $GL(N)$ spin chain. The Lax matrix of the model is $n \times n$:

$$L_{ij}^{\mathsf{RS}}(\{\dot{q}_i\}_n, \{q_i\}_n, \eta) = \frac{\eta\, \dot{q}_j}{q_i - q_j + \eta}\,. \tag{4}$$

Statement: under substitution

$$\dot{q}_j = \frac{1}{\eta} H_j\Big(\{q_i\}_n; \{\mu_i^1\}_{n_1}, \ldots, \{\mu_i^{N-1}\}_{n_{N-1}}\Big), \quad j = 1\,, ...\,, n\,,$$

where the sets $\{\mu_i^a\}$ are some solutions of BE, the eigenvalues of Lax matrix acquire the form:

$$\Big(\underbrace{g_1\,, \cdots\,, g_1}_{n-n_1}, \underbrace{g_2\,, \cdots\,, g_2}_{n_1-n_2}, \cdots\,, \underbrace{g_{N-1}\,, \cdots\,, g_{N-1}}_{n_{N-2}-n_{N-1}}, \underbrace{g_N\,, \cdots\,, g_N}_{n_{N-1}}\Big).$$

$$GL(2): \quad \Big(\underbrace{g_1\,, \cdots\,, g_1}_{\text{spin down}}, \underbrace{g_2\,, \cdots\,, g_2}_{\text{spin up}}\Big).$$

Idea of the proof. Let us introduce the following pair of matrices:

$$\mathcal{L}_{ij}(\{x_i\}_N,\{y_i\}_M,g)=\frac{g\,\hbar}{x_i-x_j+\hbar}\prod_{k\neq j}^{N}\frac{x_j-x_k+\hbar}{x_j-x_k}\prod_{\gamma=1}^{M}\frac{x_j-y_\gamma}{x_j-y_\gamma+\hbar}\,,\quad i\,,j=1\,,\ldots\,,N$$

and

$$\widetilde{\mathcal{L}}_{\alpha\beta}(\{y_i\}_M,\{x_i\}_N,g)=\frac{g\,\hbar}{y_\alpha-y_\beta+\hbar}\prod_{\gamma\neq\beta}^{M}\frac{y_\beta-y_\gamma-\hbar}{y_\beta-y_\gamma}\prod_{k=1}^{N}\frac{y_\beta-x_k}{y_\beta-x_k-\hbar}\,,\quad \alpha,\beta=1\,,\ldots\,,M$$

QC-duality is based on algebraic relation between determinants of $\mathcal{L}$ and $\widetilde{\mathcal{L}}$:

$$\det_{N\times N}\Big(\mathcal{L}\,(\{x_i\}_N,\{y_i\}_M,g)-\lambda\Big)=(g-\lambda)^{N-M}\det_{M\times M}\Big(\widetilde{\mathcal{L}}\,(\{y_i\}_M,\{x_i\}_N,g)-\lambda\Big)$$

For $M=0$ (factorization of the Ruijsenaars Lax matrix)

$$\det_{N\times N}\Big(\mathcal{L}\,(\{x_i\}_N,\{y_i\}_0,g)-\lambda\Big)=(g-\lambda)^{N}$$

$$L^{\mathsf{RS}} = \frac{1}{\eta}\begin{pmatrix} H_1 & \frac{\eta\,H_2}{q_1-q_2+\eta} & \frac{\eta\,H_3}{q_1-q_3+\eta} & \cdots & \frac{\eta\,H_n}{q_1-q_n+\eta} \\ \frac{\eta\,H_1}{q_2-q_1+\eta} & H_2 & \frac{\eta\,H_3}{q_2-q_3+\eta} & \cdots & \frac{\eta\,H_n}{q_2-q_n+\eta} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\eta\,H_1}{q_n-q_1+\eta} & \frac{\eta\,H_2}{q_n-q_2+\eta} & \frac{\eta\,H_3}{q_n-q_3+\eta} & \cdots & H_n \end{pmatrix}$$

$$\dot{q}_j = \frac{1}{\eta}H_j = g_1 \prod_{k=1}^{n} \frac{q_j - q_k + \eta}{q_j - q_k} \prod_{\gamma=1}^{n_1} \frac{q_j - \mu^1_\gamma - \eta}{q_j - \mu^1_\gamma}$$

where $\{\mu_i^a\}$ – solutions of BE. We want to prove that eigenvalues of L^{RS} are

$$\Big(\underbrace{g_1\,,\ldots\,,g_1}_{M_1},\underbrace{g_2\,,\ldots\,,g_2}_{M_2},\ldots\,,\underbrace{g_{N-1}\,,\ldots\,,g_{N-1}}_{M_{N-1}},\underbrace{g_N\,,\ldots\,,g_N}_{M_N}\Big).$$

$$L_{ij}^{(0)} = L_{ij}^{\mathsf{RS}}(\frac{1}{\eta}\{H_j^{\mathsf{XXX}}\}_n, \{q_i\}_n, \eta) = \frac{\eta\, g_1}{q_i - q_j + \eta} \prod_{k\neq j}^{n} \frac{q_j - q_k + \eta}{q_j - q_k} \prod_{\gamma=1}^{n_1} \frac{q_j - \mu_\gamma^1 - \eta}{q_j - \mu_\gamma^1}$$

$$= \mathcal{L}_{ij}(\{q_i - \eta\}_n, \{\mu_\alpha^1\}_{n_1}, g_1)\,,$$

$$L_{\alpha\beta}^{(1)} = \widetilde{\mathcal{L}}_{\alpha\beta}(\{\mu_\alpha^1\}_{n_1}, \{q_i - \eta\}_n, g_1) = \frac{\eta\, g_1}{\mu_\alpha^1 - \mu_\beta^1 + \eta} \prod_{\gamma\neq\beta}^{n_1} \frac{\mu_\beta^1 - \mu_\gamma^1 - \eta}{\mu_\beta^1 - \mu_\gamma^1} \prod_{k=1}^{n} \frac{\mu_\beta^1 - q_k + \eta}{\mu_\beta^1 - q_k}\,,$$

where $\alpha, \beta = 1\,, \ldots, n_1$.

$$\det_{n\times n}(L^{(0)} - \lambda) = (g_1 - \lambda)^{n-n_1} \det_{n_1\times n_1}(L^{(1)} - \lambda)\,.$$

Next, impose the Bethe equations

$$L_{\alpha\beta}^{(1)}\Big|_{BE_1} = \frac{\eta\, g_2}{\mu_\alpha^1 - \mu_\beta^1 + \eta} \prod_{\gamma\neq\beta}^{n_1} \frac{\mu_\beta^1 - \mu_\gamma^1 + \eta}{\mu_\beta^1 - \mu_\gamma^1} \prod_{\gamma=1}^{n_2} \frac{\mu_\beta^1 - \mu_\gamma^2 - \eta}{\mu_\beta^1 - \mu_\gamma^2}\,, \qquad \alpha, \beta = 1\,, \ldots, n_1\,,$$

i.e.,

$$L^{(1)}\Big|_{BE_1} = \mathcal{L}(\{\mu_\alpha^1 - \eta\}_{n_1}, \{\mu_\alpha^2\}_{n_2}, g_2)\,.$$

At the second step we define

$$L^{(2)}_{\alpha\beta} = \widetilde{\mathcal{L}}_{\alpha\beta}(\{\mu^2_\gamma\}_{n_2}, \{\mu^1_\gamma - \eta\}_{n_1}, g_2)\,, \qquad \alpha, \beta = 1\,, \dots, n_2\,,$$

and, similarly to the previous step, we use identity and BE to get:

$$\det_{n_1\times n_1}(L^{(1)} - \lambda) = (g_2 - \lambda)^{n_1 - n_2} \det_{n_2\times n_2}(L^{(2)} - \lambda)\,,$$

$$L^{(2)}\Big|_{BE_2} = \mathcal{L}(\{\mu^2_\alpha - \eta\}_{n_2}, \{\mu^3_\alpha\}_{n_3}, g_3)\,.$$

$$\vdots \qquad \vdots \qquad \vdots$$

and so on until the last step:

$$L^{(N-1)}_{\alpha\beta}\Big|_{BE_{N-1}} = \frac{\eta\, g_N}{\mu^{N-1}_\alpha - \mu^{N-1}_\beta + \eta} \prod_{\gamma\neq\beta}^{n_{N-1}} \frac{\mu^{N-1}_\beta - \mu^{N-1}_\gamma + \eta}{\mu^{N-1}_\beta - \mu^{N-1}_\gamma}\,, \qquad \alpha, \beta = 1\,, \dots, n_{N-1}\,.$$

The latter matrix obeys the equation $\det\limits_{n_{N-1}\times n_{N-1}}(L^{(N-1)} - \lambda) = (g_N - \lambda)^{N_{N-1}}$ which follows from determinants relation for $N = n_{N-1}$ and $M = 0$.

Similar relation between the classical Calogero-Moser and quantum Gaudin models appears in the limit $\varepsilon \to 0$ $(\eta = \nu\varepsilon)$. Then

$$L^{\mathsf{CM}}_{ij} = \lim_{\varepsilon\to 0} \frac{L^{\mathsf{RS}}_{ij} - \delta_{ij}}{\varepsilon} = \delta_{ij}\dot{q}_i + \nu \frac{1-\delta_{ij}}{q_i - q_j}\,, \qquad \dot{q}_i = p_i + \nu \sum_{k\neq i} \frac{1}{q_i - q_k}$$

$$\mathbf{H}^{\mathsf{G}}_i = \sum_{a=1}^{N} v_a E^{(i)}_{aa} + \sum_{j\neq i}^{n} \frac{\nu\,\mathbf{P}_{ij}}{q_i - q_j} = \lim_{\varepsilon\to 0} \frac{\mathbf{H}_i(\{q_i\}_n, \mathbf{g} = e^{\varepsilon v}, \varepsilon\eta) - \varepsilon\eta C^{(i)}_1}{\eta\varepsilon^2}$$

where $C^{(i)}_1 = \sum\limits_{a=1}^{N} e^{(i)}_{aa}$ и $\mathbf{P}_{ij} = \sum_{a,b=1}^{N} E^{(i)}_{ab} E^{(j)}_{ba}$

$$\mathsf{Spec}\, L^{\mathsf{CM}} \left(\tfrac{1}{\hbar} \left\{ H^{\mathsf{G}}_j \right\}_n, \left\{ q_j \right\}_n, \nu = \hbar \right) \Big|_{BE} =$$

$$= \Big(\underbrace{v_1, \ldots, v_1}_{n-n_1}, \underbrace{v_2, \ldots, v_2}_{n_1-n_2}, \ldots, \underbrace{v_{N-1}, \ldots, v_{N-1}}_{n_{N-2}-n_{N-1}}, \underbrace{v_N, \ldots, v_N}_{n_{N-1}} \Big)$$

QC-duality provides an alternative (to the nested Bethe ansatz) method for computation of spectra of the spin chains. Namely, the spectrum of the quantum transfer matrix for the inhomogeneous $\mathfrak{gl}_N$-invariant XXX spin chain on n sites with twisted boundary conditions can be found in terms of velocities of particles in the rational n-body Ruijsenaars-Schneider model. The possible values of the velocities are to be found from intersection points of two Lagrangian submanifolds in the phase space of the classical model. One of them is the Lagrangian hyperplane corresponding to fixed coordinates of all n particles and the other one is an n-dimensional Lagrangian submanifold obtained by fixing levels of n classical Hamiltonians in involution. The latter are determined by eigenvalues of the twist matrix and occupation numbers.

To find the spectrum of chain we need to find velocities of RS particles for a fixed set of eigenvalues of L^{RS}. The simplest examples show that there are more solutions then Bethe ansatz provides. It happens because the statement of QC-duality is valid for all $N+1$ SUSY chains with groups $GL(a|b)$, $a+b=N$:

$$GL(N|0)\,, GL(N-1|1)\,, \dots\,, GL(0|N)$$

The Bethe ansatz in SUSY case is slightly more complicated, but the statement is the same for any chain from the set.
Inverse problem: find velocities from fixed set of eigenvalues of the Lax matrix (i.e. action variables) and coordintes q_k.

$$\det_{1\leq i,j\leq n}\left(\lambda\delta_{ij}-\frac{\eta H_i}{q_i-q_j+\eta}\right)=\prod_{a=1}^{N}(\lambda-g_a)^{M_a}\,,\quad M_a=n_{a-1}-n_a$$

$$\sum_{1\leq i_1<\ldots<i_d\leq n}H_{i_1}\ldots H_{i_d}\prod_{1\leq\alpha<\beta\leq d}\left(1-\frac{\eta^2}{(q_{i_\alpha}-q_{i_\beta})^2}\right)^{-1}=e_d(\{P_j\})\,,$$

where $P_j=\sum_a M_a g_a^k$, and e_d – elementary symmetric functions. These algebraic equations describe spectrum of all $N+1$ SUSY quantum chains.

Thus, we come to **correspondence between classical many-body systems and the set of spin chains**. The desired simplification of the nested Bethe ansatz is in fact replaced by another problem – untangling of solutions between different chains.

We have described the quantum-classical duality

What is quantum-quantum version?

And what is classical-classical version?

The quantum-quantum case is the Matsuo-Cherednik type correspondence between the quantum Knizhnik-Zamolodchikov equations associated with $GL(N)$ and n-particle quantum Ruijsenaars-Schneider model, with n being not necessarily equal to N.

qKZ-Ruijsenaars correspondence:

The quantum Knizhnik-Zamolodchikov (qKZ) equations:

$$e^{\eta\hbar\partial_{x_i}}\Big|\Phi\Big\rangle = \mathbf{K}_i^{(\hbar)}\Big|\Phi\Big\rangle, \qquad i = 1, \ldots, n$$

$$\mathbf{K}_i^{(\hbar)} = \mathbf{R}_{i\,i-1}(x_i - x_{i-1} + \eta\hbar) \ldots \mathbf{R}_{i1}(x_i - x_1 + \eta\hbar)\mathbf{g}^{(i)}\mathbf{R}_{in}(x_i - x_n) \ldots \mathbf{R}_{i\,i+1}(x_i - x_{i+1})$$

$$\mathbf{R}_{ij}(x) = \frac{x\mathbf{I} + \eta\mathbf{P}_{ij}}{x + \eta}, \quad \mathbf{R}_{ij}(x)\mathbf{R}_{ji}(-x) = \mathrm{id}$$

Solutions can be found in the form

$$\Big|\Phi\Big\rangle = \sum_{\sigma \in S_n} \Phi_\sigma \Big|e_\sigma\Big\rangle, \qquad \Big|e_\sigma\Big\rangle = e_{\sigma(1)} \otimes e_{\sigma(2)} \otimes \ldots \otimes e_{\sigma(n)},$$

where e_a are standard basis vectors in $V = \mathbb{C}^N = \bigoplus_{a=1}^N \mathbb{C}e_a$ and S_n is the symmetric group.

There is a natural weight decomposition of the Hilbert space of the spin chain:

$$\mathcal{V} = V^{\otimes n} = \bigoplus_{M_1,\dots,M_N} \mathcal{V}(\{M_a\})$$

defined by operators

$$\mathbf{M}_a = \sum_{l=1}^{n} e_{aa}^{(l)}\,, \quad [\mathbf{M}_a, \mathbf{M}_b] = [\mathbf{H}_i, \mathbf{M}_a] = 0$$

The basis vectors in $\mathcal{V}(\{M_a\})$ are $\Big|J\Big\rangle = e_{j_1} \otimes e_{j_2} \otimes \ldots \otimes e_{j_n}$, where the number of indices j_k such that $j_k = a$ is equal to M_a for all $a = 1, \ldots, N$.

Important property:

$$\sum_{i=1}^{n} \mathbf{H}_i = \sum_{i=1}^{n} \mathbf{g}^{(i)} = \sum_{a=1}^{N} g_a \mathbf{M}_a$$

Statement: Let $\Big|\Phi\Big\rangle = \sum_J \Phi_J \Big|J\Big\rangle$ be a solution of qKZ in weight subspace $\mathcal{V}(\{M_a\})$. Then for $E = \sum_{a=1}^{N} M_a g_a$ and

$$\Psi = \sum_J \Phi_J = \Big\langle\Omega\Big|\Phi\Big\rangle, \quad \Big\langle\Omega\Big| = \sum_J \Big\langle J\Big|$$

we have

$$\sum_{i=1}^{n} \prod_{j\neq i}^{n} \frac{x_i - x_j + \eta}{x_i - x_j} \Psi(x_1, \ldots, x_i + \eta\hbar, \ldots, x_n) = E\Psi(x_1, \ldots, x_n).$$

The proof is based on the property of $\mathbf{M_a}$, relation $\widetilde{\mathbf{R}}(x) = \frac{x+\eta}{x}\mathbf{R}(x) = \mathbf{I} + \frac{\eta}{x}\mathbf{P}$ and $\Big\langle\Omega\Big|\mathbf{P}_{ij} = \Big\langle\Omega\Big|$. From the latter we have

$$\Big\langle\Omega\Big|\mathbf{R}_{ij}(x) = \Big\langle\Omega\Big|\frac{x\mathbf{I} + \eta\mathbf{P}_{ij}}{x+\eta} = \Big\langle\Omega\Big|$$

and therefore, $\Big\langle\Omega\Big|\mathbf{K}_i^{(\hbar)} = \Big\langle\Omega\Big|\mathbf{K}_i^{(0)}$. Recall that in spin chain we used $\widetilde{\mathbf{R}}(x)$.

$$e^{\eta\hbar\partial_{x_i}}\Big\langle \Omega \Big| \Phi \Big\rangle = e^{\eta\hbar\partial_{x_i}}\Psi = \Big\langle \Omega \Big| \mathbf{K}_i^{(\hbar)} \Big| \Phi \Big\rangle = \Big\langle \Omega \Big| \mathbf{K}_i^{(0)} \Big| \Phi \Big\rangle.$$

Therefore, multiplying by $\displaystyle\prod_{j\neq i}^{n} \frac{x_i - x_j + \eta}{x_i - x_j}$ and summing over i, we get:

$$\sum_{i=1}^{n} \left(\prod_{j\neq i}^{n} \frac{x_i - x_j + \eta}{x_i - x_j} \right) e^{\eta\hbar\partial_{x_i}}\Psi \;=\; \sum_{i=1}^{n} \prod_{j\neq i}^{n} \frac{x_i - x_j + \eta}{x_i - x_j} \Big\langle \Omega \Big| \mathbf{K}_i^{(0)} \Big| \Phi \Big\rangle$$

$$= \sum_{i=1}^{n} \Big\langle \Omega \Big| \mathbf{H}_i \Big| \Phi \Big\rangle \;=\; \sum_{i=1}^{n} \Big\langle \Omega \Big| \mathbf{g}^{(i)} \Big| \Phi \Big\rangle \;=\; \sum_{a=1}^{N} g_a \Big\langle \Omega \Big| \mathbf{M}_a \Big| \Phi \Big\rangle \;=\; \left(\sum_{a=1}^{N} g_a M_a \right) \Psi.$$

A natural conjecture is that Ψ is the common eigenfunction for all higher Ruijsenaars Hamiltonians $\widehat{\mathcal{H}}_k$ with the eigenvalues $E_k = \sum_a M_a g_a^k$. It appears to be true.

Classical-classical version

What is the classical analogue of the KZ equations? The answer: it is the classical Schlesinger system – non-autonomous version of the Gaudin model.

The Gaudin model:

$$L(z)=\Lambda+\sum_{i=1}^{n}\frac{S^i}{z-q_i}\,,\qquad \Lambda=\mathrm{diag}(\lambda_1,...,\lambda_N)\,,$$

The Hamiltonians follow from $\mathrm{tr}(L^2(z))$:

$$H_i=\mathrm{tr}(S^i\Lambda)+\sum_{j:j\neq i}^{n}\frac{\mathrm{tr}(S^iS^j)}{q_i-q_j}\,,\qquad i=1,...,n\,.$$

Equations of motion

$$\partial_{t_i}S^j=\frac{[S^i,S^j]}{q_i-q_i}\,,\ i\neq j\,,\qquad \partial_{t_i}S^i=-[S^i,\Lambda]-\sum_{k\neq i}^{n}\frac{[S^i,S^k]}{q_i-q_k}\,,\ i\neq j$$

are written in the Lax form

$$\partial_{t_i}L(z)=[L(z),M_i(z)]\,,\qquad M_i(z)=-\frac{S^i}{z-q_i}\,.$$

The Schlesinger system is obtained by replacing time variables with the positions of marked points q_i:

$$\partial_{q_i} S^j = \frac{[S^i, S^j]}{q_i - q_i}\,,\ i \neq j\,, \qquad \partial_{q_i} S^i = -[S^i, \Lambda] - \sum_{k \neq i}^{n} \frac{[S^i, S^k]}{q_i - q_k}\,,\ i \neq j$$

The equations of motion are equivalent to the monodromy preserving equations

$$\partial_{q_i} L(z) - \partial_z M_i(z) = [L(z), M_i(z)]$$

with the same Lax pairs as in Gaudin case.

The quantization of the Gaudin Hamiltonians is given as follows:
$(\mathrm{tr}(S^i S^j) = \sum_{a,b} S^i_{ab} S^j_{ba} \to \sum_{ab} e^i_{ab} e^j_{ba} = P_{ij})$

$$\mathbf{H}_i = \sum_{c=1}^{l} \lambda_c e^{(i)}_{cc} + \sum_{j:j \neq i}^{k} \frac{P_{ij}}{q_i - q_j}\,, \quad i = 1, ..., k\,.$$

For the quantum Schlesinger system we have non-stationary Schrodinger equations

$$(\kappa \partial_{q_i} - \mathbf{H}_i)\,|\Psi\rangle = 0\,, \quad i = 1, ..., k\,.$$

which are just the KZ equations.

Calogero-Moser model in the form of Gaudin-Schlesinger system:

Consider the Lax matrix of the classical Calogero-Moser model

$$L_{ij} = \delta_{ij} p_j + (1 - \delta_{ij}) \frac{\nu}{q_i - q_j} \,,$$

and its eigenvalue problem

$$L\Psi = \Psi\Lambda \,, \quad \Lambda = \mathrm{diag}(\lambda_1, ..., \lambda_n) \,,$$

where Ψ – is a matrix of eigenvectors. Introduce "fictitious"spectral parameter z through the gauge transformation:

$L_{ij} \to L'_{ij} = L_{ij} \frac{z-q_i}{z-q_j} = L_{ij} + L_{ij}(q_i - q_j) \frac{1}{z-q_j}$

$$L'(z) = (z - Q)L(z - Q)^{-1} = L + [L, Q](z - Q)^{-1} = L - \mathcal{O}(z - Q)^{-1},$$

where $Q = \mathrm{diag}(q_1, ..., q_n)$ and $\mathcal{O} = [Q, L]$: $\mathcal{O}_{ij} = \nu(1 - \delta_{ij})$.

$$L'(z) = L - \sum_{a=1}^{n} \frac{\mathcal{O}^a}{z - q_a} \,, \quad \mathcal{O}^a_{ij} = \nu(1 - \delta_{ij})\delta_{aj} \,.$$

$$L''(z) = \Psi^{-1} L'(z) \Psi = \Lambda - \sum_{a=1}^{n} \frac{\Psi^{-1} \mathcal{O}^a \Psi}{z - q_a}$$

In the Schlesinger case the Lax matrix is replaced by the connection $L \to \partial_z + L$. The same gauge transformations results in

$$\partial_z + L''(z) = \partial_z + \Psi^{-1} L'(z) \Psi = \partial_z + \Lambda - \nu \sum_{a=1}^{n} \frac{\Psi^{-1} \tilde{\mathcal{O}}^a \Psi}{z - q_a},$$

where $\tilde{\mathcal{O}}^a_{ij} = \nu(1 - \delta_{ij})\delta_{aj} + \delta_{ij}\delta_{aj}$. The Hamiltonians

$$H_a = \mathrm{tr}(L\tilde{\mathcal{O}}^a) - \sum_{c \neq a} \frac{\mathrm{tr}(\tilde{\mathcal{O}}^a \tilde{\mathcal{O}}^c)}{q_a - q_c}$$

Since $\mathrm{tr}(L\tilde{\mathcal{O}}^a) = p_a$ and $\mathrm{tr}(\tilde{\mathcal{O}}^a \tilde{\mathcal{O}}^c) = \delta_{ac} + \nu(1 - \delta_{ac})$, we get

$$H_a = p_a - \sum_{c \neq a} \frac{\nu}{q_a - q_c} = \dot{q}_a$$

Symplectic form:

$$\sum_i dH_i \wedge dt_i = \sum_i dp_i \wedge dq_i$$

.

Further generalizations: other root systems for classical models.

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 - \frac{1}{2}\sum_{i\neq j}^{N}\left(\frac{g_2^2}{(q_i-q_j)^2} + \frac{g_2^2}{(q_i+q_j)^2}\right) - \sum_{i=1}^{n}\frac{g_4^2}{(2q_i)^2} - \sum_{i=1}^{n}\frac{g_1^2}{q_i^2}.$$

The Olshanetsky-Perelomov Lax matrix of size $(2n+1)\times(2n+1)$:

$$L(g_1, g_2, g_4) = \begin{pmatrix} A & B & C_1 \\ -B & -A & -C_1 \\ -C_1^T & C_1^T & 0 \end{pmatrix}$$

$$A_{ij} = \delta_{ij}\dot{q}_i + (1-\delta_{ij})\frac{g_2}{q_i - q_j}, \quad B_{ij} = (1-\delta_{ij})\frac{g_2}{q_i+q_j} + \delta_{ij}\frac{\sqrt{2}g_4}{2q_i}, \quad C_{1i} = \frac{g_1}{q_i}.$$

Conditions for coupling constants: $g_1(g_1^2 - 2g_2^2 - g_2 g_4) = 0$:

1. B_N case: $g_4 = 0$, $g_1^2 = 2g_2^2$.
2. C_N case: $g_1 = 0$.
3. D_N case: $g_1 = g_4 = 0$.

In this cases on the quantum side we deal with open spin chains (with boundaries).
The boundaries are described by reflection equations:

$$R_{12}(u_1-u_2)K_1^-(u_1)R_{12}(u_1+u_2)K_2^-(u_2)=$$

$$=K_2^-(u_2)R_{12}(u_1+u_2)K_1^-(u_1)R_{12}(u_1-u_2),$$

and the transfer-matrix is

$$\mathbf{T}(u)=\operatorname*{tr}_0(K_0^+(u)R_{01}(u-q_1)..R_{0N}(u-q_N)K_0^-(u)R_{0N}(u+q_N)..R_{01}(u+q_1)).$$

Consider K-matrices

$$K^-(u)=\begin{pmatrix} 1+\frac{\alpha}{u} & 0 \\ 0 & -1+\frac{\alpha}{u} \end{pmatrix}, \qquad K^+(u)=\begin{pmatrix} 1+\frac{\beta}{-u-\hbar} & 0 \\ 0 & -1+\frac{\beta}{-u-\hbar} \end{pmatrix},$$

The Gaudin limit of the eigenvalues and Bethe equations are of the form:

$$t^{\mathsf{G}}(z)=\sum_{i=1}^{n}\left(\frac{H_i^{\mathsf{G}}}{z-q_i}-\frac{H_i^{\mathsf{G}}}{z+q_i}\right)$$

$$H_i^{\mathsf{G}}=(\alpha-\beta)\frac{1}{q_i}+\sum_{k\neq i}^{n}\left(\frac{1}{q_i-q_k}+\frac{1}{q_i+q_k}\right)-\sum_{k=1}^{n_1}\left(\frac{1}{q_i-\mu_k}+\frac{1}{q_i+\mu_k}\right),$$

And the Bethe equations are

$$(\alpha-\beta)\frac{2}{\mu_i}+\sum_{k=1}^{n}(\frac{1}{\mu_i-q_k}+\frac{1}{\mu_i+q_k})=\frac{2}{\mu_i}+\sum_{k\neq i}^{n_1}(\frac{2}{\mu_i-\mu_k}+\frac{2}{\mu_i+\mu_k}),$$

The statement of the QC-duality is as follows: Consider the C_N Calogero-Moser Lax matrix (of size $2n \times 2n$). Plugging $H_i^{\mathsf{G}} = \dot{q}_i$ and $g_2 = \hbar$, $g_4 = \sqrt{2}\hbar(\alpha-\beta)$, $g_1 = 0$ we obtain that such matrix has all zero eigenvalues on-shell Bethe equations:

$$\det(L(0,\hbar,\sqrt{2}\hbar(\alpha-\beta))-\lambda)=\lambda^{2n}.$$

Similar statements are valid for B_N and D_N models.

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Thank you!